

Marks

Do one of the following problems. If you do more than one, your grade will be the highest of the grades for the problems. If it increases your grade, your best grade on one of these problems will be used to replace the lowest grade you received on one of the components worth 20% of your grade: either the tests or on the homework component. To receive credit for these problems, all non-arithmetic work must be shown and explained in your answer. Partial credit will not be given for any part of a problem. If your solution is only partially correct, you may resubmit more drafts of it until December 10 and until you hopefully obtain full credit for it. You will also receive full credit for a problem if you provide a reference to where it has been solved already.

1. Consider functions f that are three-times differentiable real-valued functions on $[a, b]$ with $f(a) = f(b)$.
 - 15 (a) Find such a nontrivial function with

$$\left| \int_a^{(a+b)/2} f(x) dx - \int_{(a+b)/2}^b f(x) dx \right| = \frac{(b-a)^4}{192} \max_{x \in [a,b]} |f'''(x)|.$$
 (By nontrivial, I mean a function that makes the right-hand side nonzero. A nonzero cubic function would be fine).
 - 35 (b) **Either** (i) find such a function with

$$\left| \int_a^{(a+b)/2} f(x) dx - \int_{(a+b)/2}^b f(x) dx \right| > \frac{(b-a)^4}{192} \max_{x \in [a,b]} |f'''(x)|$$
or
 (ii) prove that, for all such functions,

$$\left| \int_a^{(a+b)/2} f(x) dx - \int_{(a+b)/2}^b f(x) dx \right| \leq \frac{(b-a)^4}{192} \max_{x \in [a,b]} |f'''(x)|. \quad (\text{I believe (ii) is true}).$$
- 50 2. Consider functions $f : [0, 1] \rightarrow \mathbf{R}$ that are continuous with $\int_0^1 f^3(x) dx = 0$. Prove that $\int_0^1 f^4(x) dx \geq \frac{27}{4} \left(\int_0^1 f(x) dx \right)^4$.
- 50 3. Let $f : [0, 1] \rightarrow \mathbf{R}$ be a differentiable function with a continuous derivative such that $f(0) = f(1) = -\frac{1}{6}$. Prove that $\int_0^1 (f'(x))^2 dx \geq 2 \int_0^1 f(x) dx + \frac{1}{4}$.

Note: In problems 2 and 3, as in problem 1, you will earn 15 points just for finding a nontrivial function of the type being considered for which equality is true. In any of these problems, you will earn 50 points by finding a function of the type being considered that is a counterexample to the result you are being asked to prove. You will also earn 50 points if you can prove the result in problem 1(b), even if you can't solve problem 1(a).

- 50 4. Let $g(z) = 1 - \frac{3}{\frac{1}{1-az} + \frac{1}{1-iz} + \frac{1}{1+iz}}$ (where $i = \sqrt{-1}$). Show that the coefficients in the Taylor series expansion of g about 0 are all nonnegative if and only if $a \geq \sqrt{3}$.
- 50 5. Let f be a continuously differentiable function on $[0,1]$. Let $A = f(1)$ and let $B = \int_0^1 x^{-1/2} f(x) dx$. Evaluate $\lim_{n \rightarrow \infty} n \left[\int_0^1 f(x) dx - \sum_{k=1}^n \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2} \right) f\left(\frac{(k-1)^2}{n^2}\right) \right]$ in terms of A and B .
- 50 6. Let $f : [0, 1] \rightarrow \mathbf{R}$ be an integrable function which is continuous at 1 and let $k \geq 1$. Find the value of $\lim_{n \rightarrow \infty} \frac{1}{n^k} \int_0^1 \left(\sum_{j=1}^n j^k x^k \right) f(x) dx$.

Theorems which may be useful:

1. Taylor's Theorem: Assume f is a real-valued function on $[\alpha, \beta]$, n is a nonnegative integer, $f^{(n)}$ is continuous on $[\alpha, \beta]$ and $f^{(n+1)}(t)$ exists for all t in (α, β) . Let a and x be distinct points in $[\alpha, \beta]$ and define $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$ and $f^{(0)}(a) = f(a)$.

Then, there exists a point c between a and x with $f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$.

In the following three theorems, the integrals on the right-hand side of the inequalities are assumed to be finite. Define $\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$ and $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$.

2. Cauchy-Schwarz Inequality: $\left| \int_a^b f(x)g(x) dx \right|^2 \leq \int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx$.
3. Minkowski (or Triangle) Inequality: If $1 \leq p \leq \infty$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.
4. Holder Inequality: If $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$. (This gives the Cauchy-Schwarz inequality if $p = q = 2$).
5. Integral Mean Value Theorem: If $f : [a, b] \rightarrow \mathbf{R}$ is a continuous function and w is an integrable function that does not change sign on the interval (a, b) , then there exists a point c in (a, b) such that $\int_a^b f(t) w(t) dt = f(c) \int_a^b w(t) dt$.

In the following two theorems, f is continuous on $[a, b]$.

6. If λ_1 and λ_2 are positive constants, there exists a point c in (a, b) such that $\lambda_1 f(a) + \lambda_2 f(b) = (\lambda_1 + \lambda_2) f(c)$.
7. Average value theorem: If α_i for $1 \leq i \leq n$ are nonnegative numbers with $\sum_{i=1}^n \alpha_i = 1$ and x_i is in $[a, b]$ for $1 \leq i \leq n$, then there exists a point c in (a, b) such that $f(c) = \sum_{i=1}^n \alpha_i f(x_i)$.