Marks

This is a closed book exam. No notes are permitted.

1. Evaluate the following integrals:

8 \( \int \sin^3 x \, dx \) \quad (b) \( \int \sin^{-1} x \, dx \)

12 \( \int \frac{\sqrt{x^2 - 1}}{x^4} \, dx \) \quad (d) \( \int_1^\infty \frac{(4x + 25)}{x(x^2 + 25)} \, dx \)

16 2. Let \( R \) be the region in the first quadrant in the \( xy \)-plane which is bounded by the graph of \( y = \sqrt{x} \), the line \( y = 8 \), and the \( y \)-axis. Find the following:

(a) The area of \( R \).

(b) The volume \( V_x \) of the solid generated when \( R \) is revolved about the \( x \)-axis.

(c) The volume \( V_y \) of the solid generated when \( R \) is revolved about the \( y \)-axis.

(d) The area of the surface generated when \( R \) is revolved about the \( y \)-axis.

8 3. Let \( L \) be the length of the portion of the curve \( y = \ln x \) lying between the points \((1,0)\) and \((e,1)\). Express \( L \) as a definite integral and find \( L \) either by using the Fundamental Theorem of Calculus or by using the Trapezoid Rule with four equal subintervals.

10 4. When sugar dissolves in water, it dissolves at a rate proportional to the amount of undisolved sugar present. After 2 minutes, 75% of an initial portion of sugar is still undisolved.

(a) How long does it take for 75% of the initial portion of sugar to dissolve?

(b) After 5 minutes, there are 30 grams of undisolved sugar left. How much sugar was present originally?

12 5. Find the following limits:

(a) \( \lim_{x \to 1} \frac{\tan(\pi x)}{x^3 - 1} \) \quad (b) \( \lim_{x \to 0} \frac{x^2 e^x}{\cos(4x) - 1} \)

(c) \( \lim_{x \to \infty} x \ln \left( \frac{x + a}{x - a} \right) \) where \( a > 0 \).
6. At time $t$ days the biomass $M = M(t)$, in grams, of a cell culture is given by $M = \frac{5}{1 + 4e^{-t}}$.

(a) Find $\lim_{t \to \infty} M(t)$.

(b) Calculate $\frac{dM}{dt}$ and verify that $\frac{dM}{dt} = \frac{1}{5}M(5 - M)$.

(c) What is the maximum rate of growth of $M$?

(d) What is the average biomass of the cell culture for the time period between $t = 2$ days and $t = 10$ days?

7. A car is traveling at 30 miles per hour (44 feet per second). Once the brakes are applied, the car decelerates at a constant rate and travels 200 feet before coming to a complete stop. How much time does it take for this car to go from 30 miles per hour to a complete stop?

8. If the closed interval $[a,b]$ is partitioned into $n$ equal subintervals, each of width $\Delta x$, by the numbers $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$, then

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{x_i} \Delta x = C(b^p - a^p) \] for what constants $C$ and $p$?

9. If $\frac{df}{dx} = \frac{x^2}{x^3 + 1}$ and $f(1) = 2$, then what is $f(4)$?

10. Find the area of the region enclosed by the graphs of $f(x) = 6 - x^2$ and $g(x) = |x|$.

11. (a) Find the center, radius and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{3^n(x - 2)^n}{n + 1}$.

(b) If the series in (a) represents a function $f(x)$, find a power series representation for $f'(x)$. For what values of $x$ would this power series representation be expected to converge to $f'(x)$?

12. Consider $f(x) = \int_{0}^{x} \frac{\sin(t^3)}{t} dt$.

(a) Find the Maclaurin series for $f(x)$.

(b) How many non-zero terms of the series in (a) are needed to calculate $f(1)$ with $|\text{error}| \leq 5 \times 10^{-4}$?