Math 734 Assignment 2 Due Monday, Feb. 13 at the beginning of class:.

Marks

1. On any given day, a mouse is either healthy or ill. Of the mice who are healthy today, 85% will be healthy tomorrow. Of the mice who are ill today, 40% will be ill tomorrow. Let the number of mice who are healthy n days after May 1 be x_n and let the number of mice who are ill n days after May 1 be y_n. On May 1, there are 10 healthy mice and 90 ill mice.

4 (a) If $X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, find a matrix A such that $X_{n+1} = AX_n$.

4 (b) How many mice will be ill on May 3?

4 (c) Find the eigenvalues and a basis for each eigenspace of A.

4 (d) Use your work in part (c) to find how many mice will be healthy in the long term (when the number of infected mice approaches a steady state, or doesn't change from one week to the next)? That is, what is $\lim_{n\to\infty} x_n$? *Hint*: One way to find $\lim_{n\to\infty} x_n$ is to solve $X_{n+1} = X_n = AX_n$. Your work is part (c) will be very helpful to answer this question.

4 (e) Diagonalize the matrix A. This means, write $A = PDP^{-1}$, where P is a matrix whose columns are linearly independent eigenvectors of A and D is a diagonal matrix whose diagonal entries are the eigenvalues corresponding to these eigenvectors of A. Use this to find A^{100}.

2. Assume y_n is the number of people who own homes n years after Jan. 1, 2009 and z_n is the number of people who rent apartments n years after Jan. 1, 2009, and $\begin{bmatrix} y_{n+1} \\ z_{n+1} \end{bmatrix} = A \begin{bmatrix} y_n \\ z_n \end{bmatrix}$, where $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$.

5 (a) Write a word problem that would result in this matrix equation.

5 (b) Find a diagonal matrix D and a matrix S such that $S^{-1}AS = D$.

4 (c) Find the matrix A^k. (This means find all four entries of the kth power of A).

3 (d) Find $\lim_{k\to\infty} \begin{bmatrix} y_k \\ z_k \end{bmatrix}$ if $\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 900 \\ 100 \end{bmatrix}$. [Hint: $\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$.]

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3. The matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ has eigenvalues 1 and 4.

9 (a) Find three linearly independent eigenvectors of A.

9 (b) Let P be a matrix whose columns are linearly independent eigenvectors of A and let D be a diagonal matrix whose diagonal entries are the eigenvalues corresponding to these eigenvectors of A. Compute (i) AP and (ii) PD. Explain why these two matrix products should always be equal.

3 (c) For the matrix P in part (b), what is $P^{-1}AP$?

6 (d) Use $A = PDP^{-1}$ to find expressions for (i) A^2, (ii) A^3 and (iii) A^n involving P and D.

8 (e) Use your answer in part (d) to find a matrix S such that $S^2 = A$. (S may be defined to be a square root of A).

9 (f) Let $B = \{b_1, b_2, b_3\}$ be the basis of R^3 consisting of the three linearly independent eigenvectors of A you found in part (a). Find the coordinate vectors for each column of A in the basis B. Write a matrix product which could be used to find these coordinate vectors.

3 (g) What are the coordinate vectors for Ab_1, Ab_2 and Ab_3 in the basis B? (Use the same basis you used in part (f) from now on in this problem).

4 (h) Let T be the linear transformation defined by $T(x) = Ax$. Find the matrix for T relative to the basis B, $[T]_B$. Show that $[T]_B = P^{-1}AP$.

8 (i) Let $C = A + 7I$. Without calculating C explicitly, prove that the eigenvectors of A are also eigenvectors of C and write down the eigenvalues of C. Also, prove that the eigenvectors of A are also eigenvectors of C^2 and write down the eigenvalues of C^2.

4. Assume that in a town with 800 homes, the number of homes accessing the Internet via DSL, cable and dial-up on January 1, 2010 are 50, 250 and 500 respectively. Statistics show that, of the homes that access the Internet via DSL one year, 80% still use DSL the following year, 10% use cable and 10% use dial-up. Of the homes that access the Internet via cable one year, 80% still use cable the following year, 10% use DSL and 10% use dial-up. Of the homes that access the Internet via dial-up one year, 30% still use dial-up the following year, 40% use DSL and 30% use cable.

3 (a) What is the stochastic (or transition) matrix for this situation?

6 (b) Find the eigenvalues of the stochastic matrix you found in (a).

5 (c) How many homes would you expect to subscribe to DSL when a steady-state situation has been reached? (This means the number of homes subscribing to DSL no longer changes from year to year).

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