Definition (Multiplication of natural numbers). Let $m$ be a natural number. To multiply zero to $m$ we define $0 \times m := 0$. Suppose inductively that we have defined how to multiply $n$ to $m$. Then we can multiply $n++$ to $m$ by defining $(n++) \times m := (n \times m) + m$.

Using the Peano Axioms and elementary properties of addition, prove the following propositions.

Proposition (Multiplication is commutative). Let $n, m$ be natural numbers. Then $n \times m = m \times n$.

Lemma (Natural numbers have no zero divisors). Let $n, m$ be natural numbers. Then $n \times m = 0$ if and only if at least one of $n, m$ is equal to zero.

Proposition (Distributive law). For any natural numbers $a, b, c$, we have $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Proposition (Multiplication is associative). For any natural numbers $a, b, c$, we have $(a \times b) \times c = a \times (b \times c)$.

Proposition (Multiplication preserves order). If $a$ and $b$ are natural numbers such that $a < b$ and $c$ is positive, then $ac < bc$.

Proposition (Cancelation law). Let $a, b, c$ be natural numbers such that $ac = bc$, and suppose $c$ is nonzero. Then $a = b$.

Proposition (Euclidean algorithm). Let $n$ be a natural number, and let $q$ be a positive natural number. Then there exist natural numbers $m, r$ such that $0 \leq r < q$ and $n = mq + r$. 