Let $x$ be a real number, and for each $i \in \{1, \ldots, n\}$, let $x_i$ be a real number. The most basic inequalities are

$$x^2 \geq 0 \quad (1)$$

and

$$\sum_{i=1}^{n} x_i^2 \geq 0. \quad (2)$$

We have equality only if $x = 0$ in (1), and only if $x_i = 0$ for all $i \in \{1, \ldots, n\}$ in (2). One strategy for proving inequalities is to transform them in the form (1) or (2). This usually is a long road. So we derive some consequences equivalent to (1). With $x = a - b$, $a > 0$, and $b > 0$, we get the following equivalent inequalities:

$$a^2 + b^2 \geq 2ab \iff 2(a^2 + b^2) \geq (a + b)^2$$

$$\iff \frac{a}{b} + \frac{b}{a} \geq 2$$

$$\iff x + \frac{1}{x} \geq 2, \text{ for } x > 0$$

$$\iff \frac{a + b}{2} \leq \sqrt{\frac{a^2 + b^2}{2}}.$$

Replacing $a, b$ by $\sqrt{a}, \sqrt{b}$, we get

$$a + b \geq 2\sqrt{ab} \iff \frac{a + b}{2} \geq \sqrt{ab}$$

$$\iff \sqrt{ab} \geq \frac{2ab}{a + b}.$$

In particular, we have the following famous inequality chain, called the harmonic-geometric-arithmetic-quadratic mean inequality.

**Theorem** (The HM-GM-AM-QM Inequality). Let $a, b > 0$. Then

$$\min(a, b) \leq \frac{2ab}{a + b} \leq \sqrt{ab} \leq \frac{a + b}{2} \leq \sqrt{\frac{a^2 + b^2}{2}} \leq \max(a, b).$$

By repeated use of the inequalities above, we can already prove a huge number of other inequalities.
1. For all $x$, \[
\frac{x^2 + 2}{\sqrt{x^2 + 1}} \geq 2.
\]

2. For $a, b, c \geq 0$, we have \[
(a+b)(b+c)(c+a) \geq 8abc.
\]

3. If $a_i > 0$ for all $i \in \{1, \ldots, n\}$ and $a_1a_2 \cdots a_n = 1$, then \[
(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n.
\]

4. For $a, b, c, d \geq 0$, we have \[
\sqrt{(a+b)(b+c)(c+a)} \geq \sqrt{ab} + \sqrt{cd}.
\]

5. For all $a, b, c \in \mathbb{R}$, \[
a^2 + b^2 + c^2 \geq ab + bc + ca.
\]

6. For all $a, b, c \in \mathbb{R}$, \[
\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.
\]

7. Inequalities for the sides $a, b, c$ of a triangle are very popular in competitions. The triangle inequality plays a central role. The triangle inequality occurs in four equivalent forms. Prove that these forms are equivalent.

(a) If $a, b, c$ are the sides of a triangle, then \[a + b > c.\]

(b) If $a, b, c$ are the sides of a triangle, then \[a > |b - c|.\]

(c) If $a, b, c$ are the sides of a triangle, then \[(a + b - c)(b + c - a)(c + a - b) > 0.\]

(d) If $a, b, c$ are the sides of a triangle, then there exist $x, y, z$ positive such that $a = y + z$, $b = z + x$, and $c = x + y$.

8. For all $a, b, c$, \[a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(b + c) + ca(c + a).\]

9. For $i \in \{1, \ldots, n\}$, let $a_i$ be a real number. For all real $x$, we have \[
\sum_{i=1}^{n} (a_ix + b_i)^2 = x^2 \sum_{i=1}^{n} a_i^2 + 2x \sum_{i=1}^{n} a_ib_i + \sum_{i=1}^{n} b_i^2.
\]

10. The sum $S = a_1b_1 + \ldots + a_nb_n$ is maximal (among all permutations) if the two sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are sorted the same way. $S$ is minimal if the two sequences are sorted oppositely.

11. Prove the AM-GM inequality for $n$ numbers: suppose $x_i > 0$ for all $i \in \{1, \ldots, n\}$, \[
\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}. \tag{3}
\]
12. If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are similarly sorted sequences, then
\[
\frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} \leq \frac{a_1 b_1 + \cdots + a_n b_n}{n}.
\]

13. For all $a, b, c \in \mathbb{R}$, $a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a$.

14. Find the minimum of $\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x}$, for $0 < x < \pi/2$.

15. For all $a, b, c \in \mathbb{R}$,
\[
a^4 + b^4 + c^4 \geq a^2 bc + b^2 ca + c^2 ab.
\]

16. Let $x_1, \ldots, x_n$ be positive real numbers. Show that
\[
x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} \geq x_1 x_2 \cdots x_n (x_1 + \cdots + x_n).
\]