Remark. In what follows, we make assume the slightly annoying convention that \( \mathbb{N} = \{1, 2, 3, \ldots\} \). Technically, everything that we will say holds true when we replace \( \mathbb{N} \) by the complement of any initial segment of \( \mathbb{N} \). Keep that in mind.

**Theorem** (Principle of Strong Induction). Let \( P(n) \) be a propositional function. Suppose that the following two conditions hold:

(a) \( P(1) \) is true.

(b) If \( P(i) \) is true for all \( i \leq k \), then \( P(k + 1) \) is true.

Then for every \( n \in \mathbb{N} \), \( P(n) \) is true.

What follows is a list of statements that, if time allows, we will prove in class using strong induction:

**Fundamental Theorem of Arithmetic:** Any natural number bigger than 1 can be written as a product of primes.

**An Easy Version of the Game of Nim:** Two players—Player I and Player II—move alternately in a game that starts with two equal-sized piles of coins. A move consists of removing some positive number of coins from one pile. The winner is the player who removes the last coin. Show that Player II has a winning strategy.

**The Coin-Removal Problem:** Let a string be a row of coins without gaps and without other coins beyond the ends. We write a string as a list of Hs and Ts. When we remove an H, we leave a gap (marked by a dot), and we flip all of the (at most two) coins next to it that remain. Thus \( HHT \) becomes \( T.H \) when we remove the \( H \) in the middle, and then we get \( T.. \) when we remove the new \( H \). Removing a coin from a string leaves two strings except when we remove the end. Can you guess what condition must a string satisfy in order for it to be possible to empty it (to remove all of its coins) using only the rule above described? Use strong induction to prove that the condition you found is sufficient.

**Well-Ordering Property:** Every nonempty subset of \( \mathbb{N} \) has a least element.

**The Method of Descent:** Let \( P(n) \) be a propositional function. Assume that if \( P(n) \) is false, then there exists \( k < n \) such that \( P(k) \) is false. Then \( P(n) \) is true for all \( n \in \mathbb{N} \).

**Application of Descent:** Every natural number \( n \) can be expressed in exactly one way as the product of an odd number and a power of 2.
A Tournament of the Towns Problem: For any natural number \( N \), prove the inequality

\[
\sqrt[2]{2} \sqrt[3]{3} \sqrt[4]{4} \cdots \sqrt{N - 1} \sqrt[\sqrt{N}]{} < 3.
\]

A Very Nice Problem: Let \( n \) points be selected along a circle and labeled by \( a \) or \( b \). Prove that there are at most \( \lfloor (3n + 4)/2 \rfloor \) chords which join differently labeled points and which do not intersect inside or on the circle.

The following are suggested exercises:

1. Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

2. Pick’s Theorem: Prove that the area of a simple lattice polygon (a polygon with lattice points as vertices whose sides do not cross) is given by \( I + \frac{1}{2} B - 1 \), where \( I \) and \( B \) denote respectively the number of interior and boundary lattice points of the polygon.

3. Two players—Player I and Player II—alternately name dates. On each move, a player can increase the month or the day of the month but not both. The starting position is January 1, and the player who names December 31 wins. According to the rules, the first player can start by naming some day in January after the first or the first of some month after January. Determine a winning strategy for the Player I. (Hint: Use strong induction to describe the “winning dates.”)