I. Instructions: Chapter 2 of Daniel Solow’s book *How to Read and Do Proofs* makes reference to a “key question” that corresponds to a statement. Read Solow’s discussion, and for each of the following statements, identify the corresponding key question. Once you’ve done that, prove each statement.

1. The sum of two odd integers is an even integer.

   Proof. Let \( m \) and \( n \) be two odd integers. There exist \( k_1, k_2 \in \mathbb{Z} \) such that \( m = 2k_1 + 1 \) and \( n = 2k_2 + 1 \). Hence \( m + n = 2(k_1 + k_2 + 1) \), and so \( m + n \) is an even integer.

2. An integer is even if and only if it is the sum of two odd integers.

   Proof. \((\Rightarrow)\) Suppose that \( n \) is an even integer. Then there exists \( k \in \mathbb{Z} \) such that \( n = 2k \). Let \( m = 2k - 1 \). Then \( m \) is an odd integer, and \( n = m + 1 \).

   \((\Leftarrow)\) This is simply the previous statement.

3. The product of two odd integers is an odd integer.

   Proof. Let \( m \) and \( n \) be two odd integers. There exist \( k_1, k_2 \in \mathbb{Z} \) such that \( m = 2k_1 + 1 \) and \( n = 2k_2 + 1 \). Hence \( mn = 2(2k_1k_2 + k_1 + k_2) + 1 \), and so \( mn \) is an odd integer.

4. An integer is even if and only if its square is even.

   Proof. \((\Rightarrow)\) Let \( n \) be an even integer. There exists \( k \in \mathbb{Z} \) such that \( n = 2k \). Hence \( n^2 = 4k^2 = 2(2k^2) \), and so \( n^2 \) is an even integer.

   \((\Leftarrow)\) (By contrapositive) Suppose that \( n \) is an odd integer. There exists \( k \in \mathbb{Z} \) such that \( n = 2k + 1 \). Hence \( n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \), and so \( n^2 \) is an odd integer.

II. Instructions: Prove the following set equalities:

1. \((A \cup B)^c = A^c \cap B^c\).

   Proof. \((\subseteq)\) Let \( x \in (A \cup B)^c \). By the definition of complement, we know that \( x \notin A \cup B \). This implies that \( x \notin A \) and \( x \notin B \) (otherwise, \( x \in A \cup B \)). Hence \( x \in A^c \) and \( x \in B^c \), and so \( x \in A^c \cap B^c \).

   \((\supseteq)\) Let \( x \in A^c \cap B^c \). Then \( x \in A^c \) and \( x \in B^c \), and so \( x \notin A \) and \( x \notin B \). Hence \( x \notin A \cup B \), which implies that \( x \in (A \cup B)^c \).
2. \( A \cap [(A \cap B^c)] = A \setminus B \).

   \[ \text{Proof.} \] Let \( x \in A \cap [(A \cap B^c)] \). Then \( x \in A \) and \( x \in A \cap B^c \). This implies that \( x \in A \) and \( x \in B^c \), and so \( x \in A \) and \( x \notin B \). Hence \( x \in A \setminus B \).

   \[ \text{(2)} \] Let \( x \in A \setminus B \). Then \( x \in A \) and \( x \notin B \), and so \( x \in A \) and \( x \in B^c \). This implies that \( x \in A \cap B^c \). But since \( x \in A \), this implies further that \( x \in A \cap [(A \cap B^c)] \). \( \square \)

3. \( A \cap [(A \cap B^c)^c] = A \cap B \).

   \[ \text{Proof.} \] Let \( x \in A \cap [(A \cap B^c)^c] \). Then \( x \in A \) and \( x \in (A \cap B^c)^c \). The latter condition implies that \( x \notin A \cap B^c \), and so \( x \in A^c \) or \( x \in B \). However, since \( x \in A \), it is impossible that \( x \in A^c \), from which we conclude that it must be the case that \( x \in B \). Therefore \( x \in A \cap B \).

   \[ \text{(2)} \] Let \( x \in A \cap B \). Then \( x \in A \) and \( x \in B \), and so \( x \notin B^c \). The latter implies that \( x \notin A \cap B^c \). This implies further that \( x \in [(A \cap B^c)^c] \), and since \( x \in A \), we conclude that \( x \in A \cap [(A \cap B^c)^c] \). \( \square \)

4. \( (A \cup B) \cap A^c = B \setminus A \).

   \[ \text{Proof.} \] Let \( x \in (A \cup B) \cap A^c \). Then \( x \in A \cup B \) and \( x \in A^c \). The fact that \( x \in A \cup B \) implies that \( x \in A \) or \( x \in B \), but the first disjunct is impossible, since \( x \in A^c \). Therefore \( x \in B \) and \( x \notin A \), and from this we conclude that \( x \in B \setminus A \).

   \[ \text{(2)} \] Let \( x \in B \setminus A \). Then \( x \in B \) and \( x \in A^c \). Since \( x \in A \), we know that \( x \in A \cup B \). Therefore \( x \in (A \cup B) \cap A^c \). \( \square \)

5. \( A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C) \).

   \[ \text{Proof.} \] Let \( x \in A \cap (B \setminus C) \). Then \( x \in A \) and \( x \in B \setminus C \). The latter implies that \( x \in B \) and \( x \notin C \). Since \( x \in A \) and \( x \in B \), we know that \( x \in A \cap B \). Since \( x \notin C \), we conclude that \( x \notin A \cap C \). Hence we have that \( x \in (A \cap B) \setminus (A \cap C) \).

   \[ \text{(2)} \] Let \( x \in (A \cap B) \setminus (A \cap C) \). Then \( x \in A \cap B \) and \( x \notin A \cap C \). Since \( x \in A \cap B \), we know that \( x \in A \) and \( x \in B \). From \( x \notin A \cap C \), we know that \( x \notin A \) or \( x \notin C \), but the first disjunct is impossible, since we have concluded already that \( x \in A \). Therefore \( x \notin C \). Since \( x \in B \) and \( x \notin C \), we have that \( x \in B \setminus C \), and from the fact that \( x \in A \) we conclude that \( x \in A \cap (B \setminus C) \). \( \square \)

III. Instructions: Solve the following problems:

1. Write down the converse of the following statement: “If a number is odd, then its square is odd.”

   \[ \text{Answer.} \] The statement above quoted has the form \((\forall n \in \mathbb{Z}) (O(n) \rightarrow O(n^2))\), where \(O(m)\) stands for the formula “\( m \) is odd.” The converse of such a statement should have the structure \((\forall n \in \mathbb{Z}) (O(n^2) \rightarrow O(n))\). The converse is therefore: “If the square of a number is odd, then the number itself is odd.” \( \square \)

2. Consider the following statement: “If \( f(x) = mx + b \) and \( x \neq y \), then \( f(x) \neq f(y) \).”

   (a) Write down the contrapositive of this statement.

   \[ \text{Answer.} \] “If \( f(x) = mx + b \) and \( f(x) = f(y) \), then \( x = y \).” \( \square \)

   (b) For what values of \( m \) is this contrapositive true?
Answer. For $m \neq 0$.

(c) Write down a proof of the contrapositive for the values of $m$ determined in the previous answer.

Proof. Suppose that $f(x) = mx + b$, with $m \neq 0$. Let $x, y \in \mathbb{R}$ be such that $f(x) = f(y)$. Then $mx + b = my + b$. Subtracting $b$ from both sides of this equation yields that $mx = my$, and dividing both sides by $m$ yields that $x = y$.

(d) Discuss why using the contrapositive here was a good idea.

Hint. How do you prove that two numbers are unequal?

3. Consider the following statement: “If $a$ is less than or equal to every number greater than $b$, then $a \leq b$.”

(a) Can you find a way to prove this statement directly?

Hint. I can’t get very far.

(b) Assume the premise and the negation of the conclusion. Can you produce a contradiction?

Hint. Now you have more information to work with. First, you know that for every $c \in \mathbb{R}$, if $b < c$ then $a \leq c$. (This is the premise of the implication we wish to prove.) And second, you assume also that it is not the case that $a \leq b$—that is to say, you assume also that $a > b$. This second assumption (the negation of the intended conclusion) gives you far more information than you had in the direct approach.

IV. Instructions: Answer the following questions:

1. Which of these statements are believable?

(a) “All of my 5-legged dogs can fly.”

Answer. Quite believable. Why?

(b) “I have no 5-legged dog that cannot fly.”

Hint. How many verifications do you need to make in order to prove that this statement is true for all instances?

(c) “Some of my 5-legged dogs cannot fly.”

Hint. How many 5-legged dogs can I have?

(d) “I have a 5-legged dog that cannot fly.”

Answer. Hopefully it is obvious that this is false.

2. Which of the following two statements implies the other, and why?

(a) There is a number $M$ such that, for every $x$ in the set $S$, $|x| \leq M$.

(b) For every $x$ in the set $X$, there is a number $M$ such that $|x| \leq M$. 

Partial Answer. The first item implies the second statement, but not vice versa. (Why?)

3. A fraternity has a rule for new members: each must always tell the truth or always lie. They know which does which. If I meet three of them on the street and they make the statements below, which ones (if any) should I believe?

- A says: “All three of us are liars.”
- B says: “Exactly two of us are liars.”
- C says: “The other two are liars.”

Hint. Like a good mathematician, do not allow contradictions.

4. Consider \( f : \mathbb{R} \to \mathbb{R} \). Let \( S \) be the set of functions defined by putting \( g \in S \) if there exist positive constants \( c, a \in \mathbb{R} \) such that \( |g(x)| \leq c|f(x)| \) for all \( x > a \). Without words of negation, state the meaning of “\( g \notin S \).”

Hint. Write down carefully the logical structure of the statement \( g \in S \), and then apply the rules of negation slowly, until you rid yourself of negations.

5. Three children are in line. From a collection of two red hats and three black hats, the teacher places a hat on each child’s head. The third child sees the hats on the first two, the second child sees the hat on the first, and the first child sees no hats. The children, who reason carefully, are told to speak out as soon as they can determine the color of the hat they are wearing. After 30 seconds, the first child correctly names the color of her hat. Which color is it, and why?

Hint. The children can be assumed to reason quite fast.