1. Abe Lincoln said, “You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can’t fool all of the people all of the time.” Write this sentence in logical notation, and find its negation.

Hint. Use the variable $x$ to denote individual persons and the variable $t$ for units of time. Let also $F(x, t)$ denote the statement: “You can fool person $x$ at time $t$.”

2. Consider the equation $x^4y + ay + x = 0$.
   (a) Show that the following statement is false. “For all $a, x \in \mathbb{R}$, there is a unique $y$ such that $x^4y + ay + x = 0$.

   Hint. A counterexample for this statement is an ordered pair $(a, x)$ for which the equation is either false or else true for more than one value of $y$.

   (b) Find the set of real numbers $a$ such that the following statement is true. “For all $x \in \mathbb{R}$, there is a unique $y$ such that $x^4y + ay + x = 0$.”

   Hint. The set you are looking for will turn out to be an interval.

3. Let $f$ be a real-valued function on a set $S$. In order to prove that the minimum value in the image of $f$ is $\beta$, two statements must be proved. Express each of these statements using quantifiers.

   Hint. The minimum of a set, if it exists, is an element of the set which is smaller than every other element of the set.

4. Let $T$ be the set of ordered pairs of positive real numbers. Define $f : T \to \mathbb{R}$ by $f(x, y) = \max \left\{ x, y, \frac{1}{x} + \frac{1}{y} \right\}$. Determine the minimum value in the image of $f$. ($\star$)

   Hint. Note that if $x \geq \frac{1}{x} + \frac{1}{y}$ or $x \geq \frac{1}{x} + \frac{1}{y}$, you can reduce the maximum by reducing the larger element among $x$ and $y$.

5. Let $f$ and $g$ be functions from $\mathbb{R}$ to $\mathbb{R}$. Determine which statements below are true. If true, provide a proof. If false, provide a counterexample.
Hint. Remember that $f$ is bounded if and only if there exists $M > 0$ such that for any $x \in \mathbb{R}$, $|x| \leq M$. Also, have in mind simple examples of bounded and unbounded functions when trying to determine whether the statements are true. First use your intuition, and then try to write down proofs or offer counterexamples.

(a) If $f$ and $g$ are bounded, then $f + g$ is bounded.
(b) If $f$ and $g$ are bounded, then $fg$ is bounded.
(c) If $f + g$ is bounded, then $f$ and $g$ are bounded.
(d) If $fg$ is bounded, then $f$ and $g$ are bounded.
(e) If both $f + g$ and $fg$ are bounded, then $f$ and $g$ are bounded.

Hint. This is actually true. Aim to show that $f^2$ and $g^2$ are bounded, and use that to conclude that $f$ and $g$ are bounded. Also, keep in mind that $f^2 + g^2 = (f + g)^2 - 2fg$.

6. For $S$ in the domain of a function $f$, let $f(S) = \{f(x) : x \in S\}$. Let $C$ and $D$ be subsets of the domain of $f$.

(a) Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.

Hint. The first line of this proof should start with: “Let $f(x) \in f(C \cap D)$. The second sentence should look like: “This implies that $x \in C \cap D$.

(b) Give an example where equality does not hold.