1. Give a sentence $P(n)$ depending on a natural number $n$, such that $P(1), P(2), \ldots, P(99)$ are all true but $P(100)$ is false. Make your sentence as simple as possible.

   \textit{Answer.} Let $P(n)$ be the statement \textquotedblleft $n < 100$\textquotedblright.

2. Let $P(n)$ be a mathematical statement depending on a natural number $n$. Suppose that $P(1)$ is false. Suppose also that whenever $P(n)$ is false, $P(n+1)$ is also false. Show that $P(k)$ is false for all $k \in \mathbb{N}$. (There is a very short proof!)

   \textit{Hint.} Let $Q(n) = \neg P(n)$. Show by induction on $n$ that $Q(n)$ is true for all $n \in \mathbb{N}$. Use this to deduced the intended conclusion.

3. Let $P(n)$ be a mathematical statement depending on an integer $n$. Suppose that $P(0)$ is true. Suppose also that whenever $P(n)$ is true, both $P(n+1)$ and $P(n-1)$ are also true. Show that $P(k)$ is true for all $k \in \mathbb{Z}$.

   \textit{Hint.} Let $Q(n) = P(n) \land P(-n)$. Apply induction on $n$ to prove that $Q(n)$ is true for all $n \in \mathbb{N}$. Use this to deduce the intended conclusion.

4. Let $P(n)$ be a mathematical statement depending on an integer $n$. Suppose that $P(0)$ is true. Suppose also that whenever $P(n)$ is true, at least one of $P(n+1)$ and $P(n-1)$ is true. For which $n \in \mathbb{Z}$ must $P(n)$ be true? (Justify your answer.)

   \textit{Hint.} Consider the statement $P(n)$ defined by $n^2 = n$.

   \textbf{Determine whether each of the following statements is true or false. If true, provide a proof. If false, provide a counterexample.}

5. For $n \in \mathbb{N}$,

   \[ \sum_{k=1}^{n} (2k + 1) = n^2 + 2n. \]

   \textit{Hint.} This is true.

6. If $P(2n)$ is true for all $n \in \mathbb{N}$ and $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$. 
Hint. This is false. Find a statement $P(n)$ that satisfies the hypothesis but such that $P(1)$ is false.

7. For $n \in \mathbb{N}$, $2n - 8 < n^2 - 8n + 17$.

Hint. This is false. Check out what happens when $n = 5$.

8. For $n \in \mathbb{N}$, $2n - 18 < n^2 - 8n + 8$.

Hint. This is true.

9. For $n \in \mathbb{N}$,

\[
\frac{2n - 18}{n^2 - 8n + 8} < 1.
\]

Hint. This is actually false.

Use induction to prove the following statements.

10. If $n \in \mathbb{N}$ and $x_1, \ldots, x_{2n+1}$ are odd integers, then $\sum_{i=1}^{2n+1} x_i$ is odd and $\prod_{i=1}^{2n+1} x_i$ is odd.

Hint. Too easy to give a hint!

11. A set of $n$ elements has $2^n$ subsets.

Hint. No hint for this one—though this one is not easy.

12. Given $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $\sum_{i=1}^{n} x = nx$.

Hint. It doesn’t get any easier than this.

13. The sum of two polynomials is a polynomial.

Hint. Let $P(n)$ be the statement “The sum of two polynomials of degree less than or equal to $n$ is a polynomial of degree less than or equal to $n$.” Also, recall that there is a standard way to write down the general form of a polynomial of degree less than or equal to $n$, namely such a polynomial is of the form $a_0 + a_1 x + \cdots + a_n x^n$ for some constants $a_0, a_1, \ldots, a_n$.

14. For $n \in \mathbb{N}$,

\[
\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n + 1)}{2}.
\]

Hint. Worked out in class.

15. For $n \in \mathbb{N}$,

\[
\sum_{k=1}^{n} k^3 = \left( \frac{n(n + 1)}{2} \right)^2.
\]

Hint. Too easy for a hint.
16. For $n \in \mathbb{N}$,

$$\left| \sum_{i=1}^{n} a_i \right| \leq \sum_{i=1}^{n} |a_i|.$$ 

Hint. The basis step should be when $n = 2$, but you are allowed to assume the triangle inequality, so there is nothing to prove for that case.

17. If $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$, then $f(1) = 0$ and $f(u^n) = nu^{n-1}f(u)$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

Hint. With $y = 1$, the hypothesis yields $f(x) = xf(1) + f(x)$. Thus $xf(1) = 0$ for all $x \in \mathbb{R}$, which requires $f(1) = 0$. 

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