Remark. Please pay special attention to how the solutions are written. Writing proofs demands attention to detail, grammar, and above all, precision. Rather than saying “$n$ has $2^n$ subsets,” note that we say “a set $S$ with $n$ elements has $2^n$ subsets.” From the viewpoint of style, omissions and abbreviations are not kosher.

1. Give a sentence $P(n)$ depending on a natural number $n$, such that $P(1), P(2), \ldots, P(99)$ are all true but $P(100)$ is false. Make your sentence as simple as possible.

   Solution. Let $P(n)$ be the statement “$n \neq 100$.” Then $P(1), \ldots, P(99)$ are true but $P(100)$ is false.

2. Let $P(n)$ be a mathematical statement depending on a natural number $n$. Suppose that $P(1)$ is false. Suppose also that whenever $P(n)$ is false, $P(n+1)$ is also false. Show that $P(k)$ is false for all $k \in \mathbb{N}$. (There is a very short proof!)

   Solution. Let $Q(n) = \neg P(n)$. We will prove, by induction on $n$, that $Q(k)$ is true for all $k \in \mathbb{N}$.

   Basis step: Since $P(1)$ is false, we have that $Q(1)$ is true, and this finishes the basis step.

   Inductive step: We assume that $Q(n)$ is true, and we wish to show that $Q(n+1)$ is also true. Since $Q(n)$ is true, $P(n)$ is false by the definition of $Q(n)$. By hypothesis, we know then that $P(n+1)$ is false. Since $Q(n+1) = \neg P(n+1)$, we have that $Q(n+1)$ is true, and this finishes the inductive step.

   By the principle of induction, we have thus established that $Q(k)$ is true for all $k \in \mathbb{N}$. But this clearly implies that $P(k)$ is false for all $k \in \mathbb{N}$.

3. Let $P(n)$ be a mathematical statement depending on an integer $n$. Suppose that $P(0)$ is true. Suppose also that whenever $P(n)$ is true, both $P(n+1)$ and $P(n-1)$ are also true. Show that $P(k)$ is true for all $k \in \mathbb{Z}$.

   Solution. Let $Q(n) = P(n) \land P(-n)$. We will prove, by induction on $n$, that $Q(k)$ is true for all $k \in \{0, 1, 2, \ldots\}$.

   Basis step: Since $P(0)$ is true, we have that $Q(0)$ is true, and this finishes the basis step.
**Inductive step:** We assume that $Q(n)$ is true, and we wish to show that $Q(n + 1)$ is also true. Since $Q(n)$ is true, both $P(n)$ and $P(-n)$ are true by the definition of $Q(n)$. By hypothesis, we know then that both $P(n+1)$ and $P(-(n+1))$ are true. Since $Q(n+1) = P(n+1) \land P(-(n+1))$, we have that $Q(n+1)$ is true, and this finishes the inductive step.

By the principle of induction, we have thus established that $Q(k)$ is true for all $k \in \{0, 1, 2, \ldots\}$. But this clearly implies that $P(k)$ is true for all $k \in \mathbb{Z}$. □

4. Let $P(n)$ be a mathematical statement depending on an integer $n$. Suppose that $P(0)$ is true. Suppose also that whenever $P(n)$ is true, at least one of $P(n + 1)$ and $P(n - 1)$ is true. For which $n \in \mathbb{Z}$ must $P(n)$ be true? (Justify your answer.)

**Solution.** This is false, and we provide a counterexample. Let $P(n)$ be the statement $n = 0 \lor n = 1$. Then $P(0)$ is true, and whenever $P(n)$ is true, it is indeed the case that at least one of $P(n + 1)$ and $P(n - 1)$ is true. □

Determine whether each of the following statements is true or false. If true, provide a proof. If false, provide a counterexample.

5. For $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} (2k + 1) = n^2 + 2n.$$  

**Solution.** Let $P(n)$ be the identity

$$\sum_{k=1}^{n} (2k + 1) = n^2 + 2n.$$  

We proceed by induction on $n$.

**Basis step:** $P(1)$ is simply the statement $3 = 1 + 2$, which is evidently true.

**Inductive step:** We assume that $P(n)$ is true, and we wish to show that $P(n + 1)$ is also true. Note that

$$\sum_{k=1}^{n+1} (2k + 1) = \sum_{k=1}^{n} (2k + 1) + (2(n + 1) + 1) = (n^2 + 2n) + (2n + 3) = (n + 1)^2 + 2(n + 1).$$

This establishes that $P(n + 1)$ is true, and finishes the inductive step.

By the principle of induction, we conclude that $P(k)$ is true for all $k \in \mathbb{N}$. □

6. If $P(2n)$ is true for all $n \in \mathbb{N}$ and $P(n)$ implies $P(n + 1)$ for all $n \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$.

**Solution.** This statement is false, and we provide a counterexample. Let $P(n)$ be the statement “$n > 1$.” Then $P(2n)$ is true no matter the value of $n$, and whenever $P(n)$ is true, it is indeed the case that $P(n + 1)$ is also true. But $P(1)$ is obviously false. □

7. For $n \in \mathbb{N}$, $2n - 8 < n^2 - 8n + 17$. 

Solution. This statement is false. The inequality holds for \( n \in \{1, 2, 3, 4\} \), but it fails for \( n = 5 \). (In fact, the inequality fails only for \( n = 5 \), since it is equivalent to \( 0 < n^2 - 10n + 25 = (n - 5)^2 \).)

8. For \( n \in \mathbb{N} \), \( 2n - 18 < n^2 - 8n + 8 \).

Solution. This is a true statement. The inequality is equivalent to \( 0 < n^2 - 10n + 26 = (n - 5)^2 + 1 \), which is positive for all \( n \in \mathbb{N} \).

9. For \( n \in \mathbb{N} \),
\[
\frac{2n - 18}{n^2 - 8n + 8} < 1.
\]

Solution. This statement is actually false. This inequality differs from that in the preceding problem when \( n^2 - 8n + 8 \leq 0 \). It is false for \( n \in \{2, 3, 4, 5, 6\} \).

Use induction to prove the following statements.

10. If \( n \in \mathbb{N} \) and \( x_1, \ldots, x_{2n+1} \) are odd integers, then \( \sum_{i=1}^{2n+1} x_i \) is odd and \( \prod_{i=1}^{2n+1} x_i \) is odd.

Solution. We will prove both statements with just one induction on \( n \). So let \( P(n) \) be the statement “\( \sum_{i=1}^{2n+1} x_i \) is odd and \( \prod_{i=1}^{2n+1} x_i \) is odd.”

Basis step: \( P(1) \) is simply the statement “the sum and product of three odd numbers are odd numbers.” The proof of that statement is left to the reader as an exercise.

Inductive step: We assume that \( P(n) \) is true, and we wish to show that \( P(n+1) \) is also true. Note that
\[
\sum_{i=1}^{2(n+1)+1} x_i = \sum_{i=1}^{n} (2n+1)x_i + x_{2n+2} + x_{2n+3},
\]
which is the sum of three odd numbers, and thus an odd number, by the basis step. On the other hand,
\[
\prod_{i=1}^{2(n+1)+1} x_i = \left( \prod_{i=1}^{n} (2n+1)x_i \right) x_{2n+2} x_{2n+3},
\]
which is the product of three odd numbers, and thus an odd number, by the basis step. This finishes the inductive step.

11. A set of \( n \) elements has \( 2^n \) subsets.

Solution. Let \( P(n) \) be the statement “a set of \( n \) elements has \( 2^n \) subsets.” We prove this by induction on \( n \) for \( n \geq 0 \).

Basis step: \( P(0) \) is simply the statement “the empty set has exactly one subset,” which is trivially true.
Inductive step: We assume that $P(n)$ is true, and we wish to show that $P(n + 1)$ is also true. Let $S$ be a set with $n + 1$ elements, and let $x \in S$. The subsets of $S$ consist of those containing $x$ and those not containing $x$. The subsets of $S$ not containing $x$ are subsets of $S \setminus \{x\}$, which is a set of $n$ elements, and so there are $2^n$ of them by the induction hypothesis. The subsets of $S$ containing $x$ consist of $x$ together with a subset of $S \setminus \{x\}$, and so there are also $2^n$ of them. Thus altogether we have $2^n + 2^n = 2^{n+1}$ subsets of $S$. Since $S$ was chosen arbitrarily, $P(n + 1)$ is true.

12. Given $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $\sum_{i=1}^{n} x = nx$.

Solution. Let $P(n)$ be the statement “$\sum_{i=1}^{n} x = nx$.” We proceed by induction on $n$.

Basis step: $P(1)$ is simply the statement $x = 1x$, which is obviously true.

Inductive step: We assume that $P(n)$ is true and wish to show that $P(n + 1)$ is also true. Note that

$$\sum_{i=1}^{n+1} x = \sum_{i=1}^{n} x + x = nx + x = (n + 1)x,$$

where the third equality follows from the induction hypothesis. This finishes the inductive step.

13. The sum of two polynomials is a polynomial.

Solution. Let $P(n)$ be the statement “the sum of two polynomials of degree $\leq n$ is a polynomial of degree $\leq n$. “ We proceed by induction on $n \geq 0$.

Basis step: $P(0)$ is simply the statement that the sum of any two constants is a constant, which is obviously true.

Inductive step: We assume that $P(n)$ is true and wish to show that $P(n + 1)$ is also true. Let $p(x) = a_0 + a_1x + \cdots + a_{n+1}x^{n+1}$ and $q(x) = b_0 + b_1x + \cdots + b_{n+1}x^{n+1}$ be two polynomials of degree $\leq n + 1$. Note that

$$p(x) + q(x) = (a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) + (a_{n+1} + b_{n+1})x^{n+1}.$$

The sum of the first two parentheses is, by the induction hypothesis, a polynomial of degree $\leq n$, and the final term makes the resulting polynomial have degree at most $n + 1$.

14. For $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n + 1)}{2}.$$

Solution. Let $P(n)$ denote the equation above displayed. We proceed by induction on $n$. 
Basis step: Note that $P(1)$ simply means that $-1 = -1^2$, which is obviously true. This finishes the basis step.

Inductive step: We assume that $P(n)$ is true and wish to show that $P(n + 1)$ is also true. Note that

$$
\sum_{k=1}^{n+1} (-1)^k k^2 = \sum_{k=1}^{n} (-1)^k k^2 + (1)^{n+1}(n + 1)^2 \\
= (-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n + 1)^2 \\
= (-1)^{n+1} \frac{2n^2 + 4n + 2 - n^2 - n}{2} \\
= (-1)^{n+1} \frac{n^2 + 3n + 2}{2} \\
= (-1)^{n+1} \frac{(n + 1)(n + 2)}{2},
$$

and this establishes the inductive step.

\[\square\]

15. For $n \in \mathbb{N}$,

$$
\sum_{k=1}^{n} k^3 = \left( \frac{n(n+1)}{2} \right)^2.
$$

Solution. Let $P(n)$ denote the equation displayed above. We proceed by induction on $n$.

Basis step: Note that $P(1)$ simply means that $1 = 1^2$, which is obviously true. This finishes the basis step.

Inductive step: We assume that $P(n)$ is true and wish to show that $P(n + 1)$ is also true. Note that

$$
\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n + 1)^3 \\
= \left( \frac{n(n+1)}{2} \right)^2 + (n + 1)^3 \\
= \frac{(n+1)(n^2 + 4n + 4)}{4} \\
= \left( \frac{(n + 1)(n + 2)}{2} \right)^2,
$$

and this establishes the inductive step.

\[\square\]

16. For $n \in \mathbb{N}$,

$$
\left| \sum_{i=1}^{n} a_i \right| \leq \sum_{i=1}^{n} |a_i|.
$$
Solution. Let $P(n)$ denote the inequality above displayed. We proceed by induction on $n$.

**Basis step:** Note that $P(1)$ is trivially true and $P(2)$ is nothing but the triangle inequality, which is obviously true. (Exercise: prove the triangle inequality!) This finishes the basis step.

**Inductive step:** We assume that $P(n)$ is true and wish to show that $P(n + 1)$ is also true. Note that

$$\left| \sum_{i=1}^{n+1} a_i \right| = \left| \sum_{i=1}^{n} a_i + a_{n+1} \right|$$

$$\leq \left| \sum_{i=1}^{n} a_i \right| + |a_{n+1}|$$

$$\leq \left( \sum_{i=1}^{n} |a_i| \right) + |a_{n+1}|$$

$$= \sum_{i=1}^{n+1} |a_i|,$$

and this establishes the inductive step.

\[\square\]

17. If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$, then $f(1) = 0$ and $f(u^n) = nu^{n-1}f(u)$ for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

Solution. With $y = 1$, the hypothesis yields $f(x) = xf(1) + f(x)$. Thus $xf(1) = 0$ for all $x \in \mathbb{R}$, which requires $f(1) = 0$.

For the second statement, to which we will refer as $P(n)$, we proceed by induction on $n$.

**Basis step:** Note that $f(x^1) = f(x) = 1x^0f(x)$, and so $P(1)$ is true. This finishes the basis step.

**Inductive step:** We assume that $P(n)$ is true and wish to show that $P(n + 1)$ is also true. Note that

$$f(x^{n+1}) = f(xx^n) = xf(x^n) + x^n f(x) = nx^{n-1}f(x) + x^n f(x) = (n + 1)x^n f(x),$$

and this establishes the inductive step.

\[\square\]