1. Prove that for any $n \in \mathbb{N}$, $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer.

2. The Coin-Removal Problem: Let a string be a row of coins without gaps and without other coins beyond the ends. We write a string as a list of $H$s and $T$s. When we remove an $H$, we leave a gap (marked by a dot), and we flip all of the (at most two) coins next to it that remain. Thus $HHT$ becomes $T.H$ when we remove the $H$ in the middle, and then we get $T..$ when we remove the new $H$. Removing a coin from a string leaves two strings except when we remove the end. Can you guess what condition must a string satisfy in order for it to be possible to empty it (to remove all of its coins) using only repeated application of the rule above described? Use strong induction to prove that the condition you found is sufficient.

3. A Tournament of the Towns Problem: For any natural number $N$, prove the inequality

$$\sqrt{2} \sqrt[3]{3} \sqrt[4]{4} \cdots \sqrt{(N-1)} \sqrt{N} < 3.$$

4. Let $n$ points be selected along a circle and labeled by $a$ or $b$. Prove that there are at most $\lfloor (3n - 4)/2 \rfloor$ chords which join differently labeled points and which do not intersect inside or on the circle.

5. Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

6. Two players—Player I and Player II—alternately name dates. On each move, a player can increase the month or the day of the month but not both. The starting position is January 1, and the player who names December 31 wins. According to the rules, the first player can start by naming some day in January after the first or the first of some month after January. Determine a winning strategy for the Player I. (Hint: Use strong induction to describe the “winning dates.”)