1. Let \( A = \{ x \in \mathbb{R} : x^2 < 4 \} \) and let \( B = \{ x \in \mathbb{R} : x < 2 \} \).
   (a) Is \( A \subseteq B \)? Justify your conclusion with a proof or a counterexample.

   \[ \text{Proof.} \] Let \( x \in A \). Then \( x^2 < 4 \), and so \(-2 < x < 2\). In particular, \( x < 2 \), which implies that \( x \in B \). Since \( x \) was chosen arbitrarily, we conclude that \( A \subseteq B \). \( \square \)

   (b) Is \( B \subseteq A \)? Justify your conclusion with a proof or a counterexample.

   \[ \text{Counterexample.} \] If \( x = -4 \), then \( x \in B \), since \( x < 2 \), but \( x \notin A \), since \( x^2 = 16 \not\approx 4 \). \( \square \)

2. Prove that if \( A \cap B = A \cap C \) and \( A^c \cap B = A^c \cap C \), then \( B = C \).

   \[ \text{Proof.} \ (\subseteq) \] Let \( x \in B \). We consider two cases:

   - \( x \in A \) In this case, \( x \in A \cap B \). By hypothesis, \( A \cap B = A \cap C \), and so \( x \in A \cap C \). In particular, \( x \in C \).
   - \( x \in A^c \) In this case, \( x \in A^c \cap B \). By hypothesis, \( A^c \cap B = A^c \cap C \), and so \( x \in A^c \cap C \). In particular, \( x \in C \).

   In any case, \( x \in C \), and because \( x \) was chosen arbitrarily, we conclude that \( B \subseteq C \).

   \[ \text{\( \supseteq \) Exercise to the reader.} \ \square \]

3. Prove that \( A \times (B \cup C) = (A \times B) \cup (A \times C) \).

   \[ \text{Proof.} \ (\subseteq) \] Let \( (x, y) \in A \times (B \cup C) \). Then \( x \in A \) and \( y \in B \cup C \). So \( x \in A \) and \( y \in B \), or \( x \in A \) and \( y \in C \). In the former case, \( (x, y) \in A \times B \), and in the latter \( (x, y) \in A \times C \). So \( (x, y) \in (A \times B) \cup (A \times C) \). Since \( (x, y) \) was chosen arbitrarily, \( A \times (B \cup C) \subseteq (A \times B) \cup (A \times C) \).

   \[ \text{\( \supseteq \) Let \( (x, y) \in (A \times B) \cup (A \times C) \). Then \( (x, y) \in A \times B \) or \( (x, y) \in A \times C \). So \( x \in A \) and \( y \in B \cup C \), and hence \( (x, y) \in A \times (B \cup C) \). Since \( (x, y) \) was chosen arbitrarily, \( (A \times B) \cup (A \times C) \subseteq A \times (B \cup C) \).} \ \square \]

4. Prove that \( A \times (B \cap C) = (A \times B) \cap (A \times C) \).
Proof. \((\subseteq)\) Let \((x, y) \in A \times (B \cap C)\). Then \(x \in A\) and \(y \in B \cap C\); we have that \((x, y) \in A \times B\), and since \(x \in A\) and \(y \in C\), we have that \((x, y) \in A \times C\). Finally, since \((x, y) \in A \times B\) and \((x, y) \in A \times B\), we have that \((x, y) \in (A \times B) \cap (A \times C)\). Because \((x, y)\) was chosen arbitrarily, the desired inclusion follows.

\((\supseteq)\) Let \((x, y) \in (A \times B) \cap (A \times C)\). Then \((x, y) \in A \times B\) and \((x, y) \in A \times C\). So \(x \in A\) and \(y \in B \cap C\). This implies that \((x, y) \in A \times (B \cap C)\). Because \((x, y)\) was chosen arbitrarily, the desired inclusion follows.

5. Prove that \(A \times (B \setminus C) = (A \times B) \setminus (A \times C)\).

Proof. \((\subseteq)\) Let \((x, y) \in A \times (B \setminus C)\). Then \(x \in A\) and \(y \in B \setminus C\), which in turn implies that \((x, y) \in A \times B\). Furthermore, since \((x, y) \in B \setminus C\), we have that \(y \notin C\), and this implies that \((x, y) \in (A \times B) \setminus (A \times C)\). Since \((x, y)\) was chosen arbitrarily, the desired inclusion follows.

\((\supseteq)\) Let \((x, y) \in (A \times B) \setminus (A \times C)\). Then \((x, y) \in A \times B\) and \((x, y) \notin A \times C\). This implies that \(x \in A\), \(y \in B\), and \(y \notin C\), which in turn implies that \((x, y) \in A \times (B \setminus C)\). Since \((x, y)\) was chosen arbitrarily, the desired inclusion follows.

6. For each natural number \(n\), let \(A_n = \{n, n+1, n+2, n+3\}\). Use the roster method to specify each of the following sets:

\(a\) \(\bigcap_{j=1}^{3} A_j\)

\(\text{Answer.} \ \bigcap_{j=1}^{3} A_j = \{3, 4\}.\)

\(b\) \(\bigcup_{j=3}^{7} A_j\)

\(\text{Answer.} \ \bigcup_{j=3}^{7} A_j = \{3, 4, 5, 6, 7, 8, 9, 10\}.\)

\(c\) \(\bigcup_{j=3}^{5} A_j\)

\(\text{Answer.} \ \bigcup_{j=3}^{5} A_j = A_3.\)

\(d\) \(A_9 \cap \left( \bigcup_{j=3}^{7} A_j \right)\)

\(\text{Answer.} \ A_9 \cap \left( \bigcup_{j=3}^{7} A_j \right) = \{9, 10\}.\)

\(e\) \(\bigcap_{j=1}^{5} A_j\)

\(\text{Answer.} \ \bigcap_{j=1}^{5} A_j = \emptyset.\)

\(f\) \(\bigcup_{j=1}^{5} (A_j \cap A_{j+1})\)

\(\text{Answer.} \ \bigcup_{j=1}^{5} (A_j \cap A_{j+1}) = \{2, 3, 4, 5, 6, 7, 8\}.\)
7. For each positive real number \( r \), define \( T_r \) to be the closed interval \([−r^2, r^2]\). That is, 
\[
T_r = \{ x \in \mathbb{R} : -r^2 \leq x \leq r^2 \}.
\]
Let \( \Lambda = \{ m \in \mathbb{N} : 1 \leq m \leq 10 \} \). Specify each of the following sets:

(a) \( \bigcup_{k \in \Lambda} T_k \)

\[ \text{Answer. } \bigcup_{k \in \Lambda} T_k = T_{10}. \]

(b) \( \bigcup_{r \in \mathbb{R}^+} T_r \)

\[ \text{Answer. } \bigcup_{r \in \mathbb{R}^+} T_r = \mathbb{R}. \]

(c) \( \bigcup_{k \in \mathbb{N}} T_k \)

\[ \text{Answer. } \bigcup_{k \in \mathbb{N}} T_k = \mathbb{R}. \]

(d) \( \bigcap_{k \in \Lambda} T_k \)

\[ \text{Answer. } \bigcap_{k \in \Lambda} T_k = T_1. \]

(e) \( \bigcap_{r \in \mathbb{R}^+} T_r \)

\[ \text{Answer. } \bigcap_{r \in \mathbb{R}^+} T_r = \{0\}. \]

(f) \( \bigcap_{k \in \mathbb{N}} T_k \)

\[ \text{Answer. } \bigcap_{k \in \mathbb{N}} T_k = \{0\}. \]

8. Let \( I \) be a nonempty index set and let \( \mathcal{A} = \{ A_i : i \in I \} \) be an indexed family of sets. Prove the following statements:

(a) For each \( j \in I \), \( A_j \subseteq \bigcup_{i \in I} A_i \).

\[ \text{Proof. } \text{Let } j \in I, \text{ and let } x \in A_j. \text{ Since } j \in I, \text{ there exists indeed } i \in I \text{ (namely } j \text{ itself) such that } x \in A_i, \text{ and so } x \in \bigcup_{i \in I} A_i. \text{ Since } x \text{ was chosen arbitrarily, } A_j \subseteq \bigcup_{i \in I} A_i. \]

(b) \( (\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} A_i^c \).

\[ \text{Proof. } (\subseteq) \text{ Let } x \in (\bigcup_{i \in I} A_i)^c. \text{ Then } x \notin \bigcup_{i \in I} A_i. \text{ This implies that for all } i \in I, \text{ } x \notin A_i. \text{ Hence } x \in \bigcap_{i \in I} A_i^c. \text{ Since } x \text{ was chosen arbitrarily, we conclude that } (\bigcup_{i \in I} A_i)^c \subseteq \bigcap_{i \in I} A_i^c. \]

\[ (\supseteq) \text{ Let } x \in \bigcap_{i \in I} A_i^c. \text{ Then for all } i \in I, \text{ } x \notin A_i. \text{ This implies that } x \notin \bigcup_{i \in I} A_i, \text{ and so } x \in (\bigcup_{i \in I} A_i)^c. \text{ Since } x \text{ was chosen arbitrarily, } \bigcap_{i \in I} A_i^c \subseteq (\bigcup_{i \in I} A_i)^c. \]

9. Let \( I \) be a nonempty index set and let \( \mathcal{A} = \{ A_i : i \in I \} \) be an indexed family of sets. Prove the following statements:

(a) \( B \cap (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} (B \cap A_i) \)
Proof. ($\subseteq$) Let $x \in B \cap \left( \bigcup_{i \in I} A_i \right)$. Then $x \in B$ and there exists $j \in I$ such that $x \in A_j$. Therefore $x \in B \cap A_j$, for some $j \in I$, and hence $x \in \bigcup_{i \in I} (B \cap A_i)$. Since $x$ was chosen arbitrarily, we conclude that $B \cap \left( \bigcup_{i \in I} A_i \right) \subseteq \bigcup_{i \in I} (B \cap A_i)$.

($\supseteq$) Let $x \in \bigcup_{i \in I} (B \cap A_i)$. Then there exists $j \in I$ such that $x \in B \cap A_j$. So $x \in B$ and, for some $j \in I$, $x \in A_j$. This readily implies that $x \in B \cap \left( \bigcup_{i \in I} A_i \right)$. Since $x$ was chosen arbitrarily, we conclude that $\bigcup_{i \in I} (B \cap A_i) \subseteq B \cap \left( \bigcup_{i \in I} A_i \right)$.

\[ \square \]

10. Let $I, J$ be nonempty index sets and let $A = \{ A_i : i \in I \}$ and $B = \{ B_j : j \in J \}$ be indexed families of sets. Use the distributive laws to:

(a) Write $\left( \bigcup_{i \in I} A_i \right) \cap \left( \bigcup_{j \in J} B_j \right)$ as a union of intersections of two sets.

\textit{Solution.} Easy as $\pi$. Write down

\[ \left( \bigcup_{i \in I} A_i \right) \cap \left( \bigcup_{j \in J} B_j \right) = \bigcup_{i \in I} \left( A_i \cap \bigcup_{j \in J} B_j \right) = \bigcup_{i \in I, j \in J} \left( A_i \cap B_j \right) = \bigcup_{(i, j) \in I \times J} \left( A_i \cap B_j \right). \]

\[ \square \]

(b) Write $\left( \bigcap_{i \in I} A_i \right) \cup \left( \bigcap_{j \in J} B_j \right)$ as an intersection of unions of two sets.

\textit{Solution.} Easy as $\pi$. Write down

\[ \left( \bigcap_{i \in I} A_i \right) \cup \left( \bigcap_{j \in J} B_j \right) = \bigcap_{i \in I} \left( A_i \cup \bigcap_{j \in J} B_j \right) = \bigcap_{i \in I, j \in J} \left( A_i \cup B_j \right) = \bigcap_{(i, j) \in I \times J} \left( A_i \cup B_j \right). \]

\[ \square \]
11. Let $I$ be a nonempty index set and let $A = \{A_i : i \in I\}$ be an indexed family of sets. Also, assume that $J \subseteq I$ and $J \neq \emptyset$. Prove that

(a) $\bigcup_{j \in J} A_j \subseteq \bigcup_{i \in I} A_i$.

Proof. Let $x \in \bigcup_{j \in J} A_j$. Then there exists $j \in J$ such that $x \in A_j$. Since $J \subseteq I$, we have that $j \in I$, and so there exists $i \in I$ such that $x \in A_j$. This in turn implies that $x \in \bigcup_{i \in I} A_i$. Since $x$ was chosen arbitrarily, we conclude that $\bigcup_{j \in J} A_j \subseteq \bigcup_{i \in I} A_i$. □

(b) $\bigcap_{i \in I} A_i \subseteq \bigcap_{j \in J} A_j$.

Proof. Let $x \in \bigcap_{i \in I} A_i$. Then for all $i \in I$, $x \in A_i$. Since $J \subseteq I$, we have that for all $j \in J$, $x \in A_j$. This in turn implies that $x \in \bigcap_{j \in J} A_j$. Since $x$ was chosen arbitrarily, we conclude that $\bigcap_{i \in I} A_i \subseteq \bigcap_{j \in J} A_j$. □

12. For each natural number $n$, let $A_n = \{x \in \mathbb{R} : n < x < n + 1\}$. Prove that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint family of sets and that $\bigcup_{n \in \mathbb{N}} A_n = (\mathbb{R} \setminus \mathbb{N})$.

Proof. We first prove that given $m \neq n$, $A_m \cap A_n = \emptyset$. By definition, we know that $A_m = ]m, m+1[$ and $A_n = ]n, n+1[$. Since $m \neq n$, we have two possibilities: $m < n$ and $n < m$. Without loss of generality, we may assume that $m < n$. If $m < n$, then $m+1 \leq n$, and so $m+1 \notin ]n, n+1[$. Moreover, if $x \in ]m, m+1[$, then $x < m+1 \leq n$ and so $x \notin ]n, n+1[$. This shows that $A_m \cap A_n \subseteq \emptyset$. Since $\emptyset$ is a subset of any set, both inclusions are established, and $A_m \cap A_n = \emptyset$.

We now proceed to show that $\bigcup_{n \in \mathbb{N}} A_n = (\mathbb{R} \setminus \mathbb{N})$.

(⊆) Let $x \in \bigcup_{n \in \mathbb{N}} A_n$. Then there exists $n \in \mathbb{N}$ such that $x \in A_n$. So $x \in ]n, n+1[$, and hence $x \in \mathbb{R} \setminus \mathbb{N}$. Since $x$ was chosen arbitrarily, we have established that $\bigcup_{n \in \mathbb{N}} A_n \subseteq (\mathbb{R} \setminus \mathbb{N})$.

(⊇) Let $x \in \mathbb{R} \setminus \mathbb{N}$. Then $x \in ]\lfloor x \rfloor, \lfloor x \rfloor + 1[ = A_{\lfloor x \rfloor}$. Therefore $x \in \bigcup_{n \in \mathbb{N}} A_n$. Since $x$ was chosen arbitrarily, we conclude that $(\mathbb{R} \setminus \mathbb{N}) \subseteq \bigcup_{n \in \mathbb{N}} A_n$. □

13. For each natural number $n$, let $A_n = \{k \in \mathbb{N} : k \geq n\}$. Determine if the following statements are true or false. Justify each conclusion with a proof or counterexample, as appropriate.

(a) For all $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j \neq \emptyset$.

Proof. Let $i, j \in \mathbb{N}$ with $i \neq j$. Then either $i < j$ or $j < i$. Without loss of generality, let us assume that $i < j$. Then $A_i \cap A_j = A_j \neq \emptyset$. □

(b) $\bigcap_{k \in \mathbb{N}} A_k = \emptyset$.

Proof. We know that $\emptyset$ is a subset of any set, so all we need to show is the inclusion $\bigcap_{k \in \mathbb{N}} A_k \subseteq \emptyset$. Let $x \in \bigcap_{k \in \mathbb{N}} A_k$. Then $x \in A_k$ for all $k \in \mathbb{N}$. However, $\lfloor x \rfloor + 1 \in \mathbb{N}$ and $x \notin A_{\lfloor x \rfloor + 1}$, a contradiction. □

14. Give an example of an indexed family of sets $\{A_n : n \in \mathbb{N}\}$ such that all three of the following conditions are true:

(a) For each $m \in \mathbb{N}$, $A_m \subseteq ]0, 1[$;
(b) For each $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j \neq \emptyset$; and
(c) $\bigcap_{k \in \mathbb{N}} A_k = \emptyset.$

**Answer.** Let $A_0 = ]0, 1[,$ and for $m \in \mathbb{N} \setminus \{0\},$ define $A_m := ]0, \frac{1}{m}[,.$ Then for each $m \in \mathbb{N},$ $A_m \subseteq ]0, 1[.$ Also, for each $i, j \in \mathbb{N},$ if $i \neq j,$ then $A_i \cap A_j = A_{\min(i,j)} \neq \emptyset.$ Finally, $\bigcap_{k \in \mathbb{N}} A_k = \emptyset,$ and the verification of this fact is left as an exercise to the reader. \qed