1. For which sets $A$ is it the case that the only bijection from $A$ to $A$ is the identity?

2. Let $A$ be the set of the days of the week. Let $f$ assign to each day of the week the number of letters in its name in English. Is $f$ an injection?

3. Let $A$ be the set of subsets of $\{1, 2, \ldots, n\}$ that have even size, and let $B$ be the set of subsets of $\{1, 2, \ldots, n\}$ that have odd size. Establish a bijection from $A$ to $B$. ($\star$)

4. Verify that $f(x) = \frac{2x-1}{2x(1-x)}$ defines a bijection from the interval $]0, 1[$ to $\mathbb{R}$. (Hint: In the proof that $f$ is surjective, use the quadratic formula.) ($\star$)

5. Let $f$ and $g$ be surjections from $\mathbb{R}$ to $\mathbb{R}$, and let $h = fg$ be their product. Must $h$ also be surjective? Give a proof or a counterexample.

6. Determine which formulas below define surjections from $\mathbb{Z}_+ \times \mathbb{Z}_+$ onto $\mathbb{Z}_+$:

(a) $f(a, b) = a + b$.
(b) $f(a, b) = ab$.
(c) $f(a, b) = ab(b + 1)/2$.
(d) $f(a, b) = (a + 1)b(b + 1)/2$.
(e) $f(a, b) = ab(a + b)/2$.

7. Given $f : \mathbb{R} \to \mathbb{R}$, suppose that there are positive constants $c, \alpha$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \geq c|x - y|^\alpha$. Prove that $f$ is injective.

8. If $f : A \to B$ and $g : B \to C$ are bijections, prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

9. Given $f : A \to B$ and $g : B \to C$, let $h = g \circ f$. Determine which of the following statements are true. Give proofs for the true statements and counterexamples for the false statements.

(a) If $h$ is injective, then $f$ is injective.
(b) If $h$ is injective, then $g$ is injective.
(c) If $h$ is surjective, then $f$ is surjective.
(d) If $h$ is surjective, then $g$ is surjective.

10. Consider $f : A \to B$ and $g : B \to A$. Answer each question below by providing a proof or a counterexample.

(a) If $f(g(y)) = y$ for all $y \in B$, does it follow that $f$ is a bijection?
(b) If \( g(f(x)) = x \) for all \( x \in A \), does it follow that \( f(g(y)) = y \) for all \( y \in B \)?

11. Consider \( f : A \to B \) and \( g : B \to A \). Prove that if \( f \circ g \) and \( g \circ f \) are identity functions, then \( f \) is a bijection. In particular, prove that

(a) If \( f \circ g \) is the identity function on \( B \), then \( f \) is surjective.

(b) If \( g \circ f \) is the identity function on \( A \), then \( f \) is injective.

12. Consider \( f : A \to A \). Prove that if \( f \circ f \) is injective, then \( f \) is injective.

13. Construct an explicit bijection from the closed interval \([0, 1]\) onto the open interval \([0, 1] \). (⋆)

14. Consider the function \( g : \mathbb{R} \to \mathbb{R} \) defined by

\[
g(x) = \frac{1}{1 + 2|x| - x}
\]

where for any \( x \in \mathbb{R}, \lfloor x \rfloor \) denotes the largest natural number less than or equal to \( x \). (For example, \( \lfloor \pi \rfloor = 3, \lfloor 3 \rfloor = 3, \) and \( \lfloor 3.99999 \rfloor = 3 \).) Let \( f : \mathbb{N} \to \mathbb{Q} \) be the function defined by the following rule:

\[
f(n) = \underbrace{(g \circ \cdots \circ g)}_{n+1\text{-times}}(0).
\]

Then prove that \( f \) is a bijection between \( \mathbb{N} \) and \( \mathbb{Q}_+ \), where \( \mathbb{Q}_+ \) denotes the set \( \{x \in \mathbb{Q} : x \geq 0\} \).

(⋆⋆⋆) (At least convince yourself that this is true!)