1. For which sets $A$ is it the case that the only bijection from $A$ to $A$ is the identity?

*Answer.* For sets $A$ consisting of exactly one element, as well as for the empty set.

2. Let $A$ be the set of the days of the week. Let $f$ assign to each day of the week the number of letters in its name in English. Is $f$ an injection?

*Answer.* No. $f(\text{Monday}) = f(\text{Sunday})$ but Monday $\neq$ Sunday.

3. Let $A$ be the set of subsets of $\{1, 2, \ldots, n\}$ that have even size, and let $B$ be the set of subsets of $\{1, 2, \ldots, n\}$ that have odd size. Establish a bijection from $A$ to $B$. ($\star$)

*Proof.* Let $A$ be the collection of even-sized subsets of $\{1, 2, \ldots, n\}$, and let $B$ be the collection of odd-sized subsets. For each $x \in A$, define $f(x)$ as follows:

$$f(x) = \begin{cases} x \setminus \{n\} & \text{if } n \in x \\ x \cup \{n\} & \text{if } n \notin x. \end{cases}$$

By this definition, $\#(x)$ and $\#(f(x))$ differ by one, so $f(x)$ is a set of odd size and $f$ maps $A$ into $B$.

We claim first that $f$ is one-to-one. Consider distinct $x, y \in A$. If both contain or both omit $n$, then $f(x)$ and $f(y)$ agree on whether they contain $n$ but differ outside $\{n\}$. If exactly one of $\{x, y\}$ contains $n$, then exactly one of $\{f(x), f(y)\}$ contains $n$. Thus $f(x) \neq f(y)$, and $f$ is one-to-one.

We next claim that $f$ is onto. If $z \in B$, then flipping whether $n$ is present in $z$ yields a subset $x$ such that $f(x) = z$, so $f$ is also onto, and hence $f$ is a bijection.

4. Verify that $f(x) = \frac{2x-1}{2x(1-x)}$ defines a bijection from the interval $]0, 1[$ to $\mathbb{R}$. (Hint: In the proof that $f$ is surjective, use the quadratic formula.) ($\star$)

*Proof.* $f$ is *injective* Suppose that $f(x) = f(y)$. Then we obtain that

$$(2x - 1)2y(1 - y) = (2y - 1)2x(1 - x),$$

which simplifies further to $2(y^2 - x^2) - 2(y - x) = 4xy(y - x)$. If $y \neq x$, then we can divide by $2(y - x)$ to obtain $y + x - 1 = 2xy$. Rewriting this as $-xy = (x - 1)(y - 1)$ makes it clear that there is no solution when $x, y \in ]0, 1[$, since the left side is negative and the right side is positive. Therefore we conclude that $x = y$, and so $f$ is injective.
Suppose that $f(x) = b$. We solve for $x$ to obtain $x \in [0, 1]$ such that $f(x) = b$. Observe that $b = 0$ is achieved by $x = \frac{1}{2}$, so we may assume that $b \neq 0$. Clearing fractions leads to $xb - x^2b = x - \frac{1}{2}$, or $bx^2 + (1 - b)x - \frac{1}{2} = 0$. The quadratic formula yields

$$x = \frac{b - 1 \pm \sqrt{b^2 + 1}}{2b}.$$ 

The magnitude of the square root is larger than $|b|$. Therefore, choosing the negative sign in the numerator yields a negative $x$, which is not in the domain of $f$. We therefore choose the positive sign.

If $b > 0$, then the square root is less than $b + 1$, and we obtain $x < \frac{b - 1 + b + 1}{2b} = 1$. Also, the square root is bigger than 1, so $x > 0$. If $b < 0$, then let $b' = -b$. The formula for $x$ becomes $x = \frac{b' + 1 - \sqrt{b'^2 + 1}}{2b'}$, where $b' > 0$. The square root is strictly between 1 and $b' + 1$, so $x$ is strictly between $\frac{1}{2}$ and 0. In each case, we have found $x$ in the domain $[0, 1]$ such that $f(x) = b$, and we have thus established that $f$ is onto.

5. Let $f$ and $g$ be surjections from $\mathbb{R}$ to $\mathbb{R}$, and let $h = fg$ be their product. Must $h$ also be surjective? Give a proof or a counterexample.

Counterexample. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = g(x) = x$. Then both $f$ and $g$ are surjective, but $(f \cdot g)(x) = x^2$ is not surjective.

6. Determine which formulas below define surjections from $\mathbb{Z}_+ \times \mathbb{Z}_+$ onto $\mathbb{Z}_+$:

(a) $f(a, b) = a + b$.

Answer. Not surjective. (Why?)

(b) $f(a, b) = ab$.

Answer. Surjective. (Why?)

(c) $f(a, b) = ab(b + 1)/2$.

Answer. Surjective. (Why?)

(d) $f(a, b) = (a + 1)b(b + 1)/2$.

Answer. Not surjective. (Why?)

(e) $f(a, b) = ab(a + b)/2$.

Answer. Not surjective (Why?)

7. Given $f : \mathbb{R} \rightarrow \mathbb{R}$, suppose that there are positive constants $c, \alpha$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \geq c|x - y|^\alpha$. Prove that $f$ is injective.

Proof. Suppose that $f$ is not injective. Then there exist distinct numbers $x$ and $y$ such that $f(x) = f(y)$. Since $c|x - y|^\alpha > 0$, this contradicts the hypothesis.

8. If $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections, prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. 
Proof. We use exercise 11. Since $f$ and $g$ are bijections, both are invertible. Note that
\[
(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1}
\]
\[
= g \circ \text{Id}_B \circ g^{-1}
\]
\[
= g \circ g^{-1}
\]
\[
= \text{Id}_C
\]
and
\[
(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f
\]
\[
= f^{-1} \circ \text{Id}_C \circ f
\]
\[
= f^{-1} \circ f
\]
\[
= \text{Id}_A.
\]

9. Given $f : A \to B$ and $g : B \to C$, let $h = g \circ f$. Determine which of the following statements are true. Give proofs for the true statements and counterexamples for the false statements.

(a) If $h$ is injective, then $f$ is injective.

Answer. True. (Why?)

(b) If $h$ is injective, then $g$ is injective.

Answer. False. (Why?)

(c) If $h$ is surjective, then $f$ is surjective.

Proof. False. (Why?)

(d) If $h$ is surjective, then $g$ is surjective.

Proof. True. (Why?)

10. Consider $f : A \to B$ and $g : B \to A$. Answer each question below by providing a proof or a counterexample.

(a) If $f(g(y)) = y$ for all $y \in B$, does it follow that $f$ is a bijection?

Proof. No. (Why?)

(b) If $g(f(x)) = x$ for all $x \in A$, does it follow that $f(g(y)) = y$ for all $y \in B$?

Proof. No. (Why?)

11. Consider $f : A \to B$ and $g : B \to A$. Prove that if $f \circ g$ and $g \circ f$ are identity functions, then $f$ is a bijection. In particular, prove that

(a) If $f \circ g$ is the identity function on $B$, then $f$ is surjective.

Proof. If $y \in B$ then $f(g(y)) = y$. Hence there is an element of $A$ mapped to $y$ by $f$; namely, the element $g(y)$. This shows that $f$ satisfies the definition of a surjection.
(b) If $g \circ f$ is the identity function on $A$, then $f$ is injective.

Proof. If $x \in A$, we have that $g(f(x)) = x$. If $f$ is not injective, then there exist distinct elements $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. If we apply $g$ to both sides of the equality, we obtain $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$, which contradicts our choice of distinct elements. Hence our assumption that $f$ is not injective must be wrong.

12. Consider $f : A \to A$. Prove that if $f \circ f$ is injective, then $f$ is injective.

Proof. Suppose that $f(x) = f(y)$. Then $f(f(x)) = f(f(y))$. By the definition of composition, this yields $(f \circ f)(x) = (f \circ f)(y)$. The hypothesis that $f \circ f$ is injective yields now that $x = y$. We have proved that $f(x) = f(y)$ implies $x = y$, and so $f$ is injective.

13. Construct an explicit bijection from the closed interval $[0, 1]$ onto the open interval $]0, 1[$. ($\star$)

Hint. This is a very nice problem. In short, no hint!

14. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \frac{1}{1 + 2[x] - x}$$

where for any $x \in \mathbb{R}$, $[x]$ denotes the largest natural number less than or equal to $x$. (For example, $[\pi] = 3$, $[3] = 3$, and $[3.99999] = 3$.) Let $f : \mathbb{N} \to \mathbb{Q}$ be the function defined by the following rule:

$$f(n) = (g \circ \cdots \circ g)(0).$$

Then prove that $f$ is a bijection between $\mathbb{N}$ and $\mathbb{Q}_+$, where $\mathbb{Q}_+$ denotes the set $\{x \in \mathbb{Q} : x \geq 0\}$.

($\star \star \star$) (At least convince yourself that this is true!)

Remark. This last exercise should be a final project.