RANGES OF POSITIVE CONTRACTIVE PROJECTIONS IN NAKANO SPACES

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Abstract. We show that in any nontrivial Nakano space $X = L_{p(\cdot)}(\Omega, \Sigma, \mu)$ with essentially bounded random exponent function $p(\cdot)$, the range $Y = R(P)$ of a positive contractive projection P is itself representable as a Nakano space $L_{p_Y(\cdot)}(\Omega_Y, \Sigma_Y, \nu_Y)$, for a certain measurable set $\Omega_Y \subseteq \Omega$ (the support of the range), a certain sub-sigma ring $\Sigma_Y \subseteq \Sigma$ (with maximal element Ω_Y) naturally determined by the lattice structure of Y, and a semi-finite measure ν_Y, namely the restriction of some measure ν on Σ which is equivalent to μ. Furthermore, we show that the random exponent $p_Y(\cdot)$ associated with such a range can be taken to be the restriction to Ω_Y of the random exponent $p(\cdot)$ (this restriction turns out to be Σ_Y-measurable). As an application of this result, we find Banach lattice isometric characterizations of suitable classes of Nakano spaces. These classes are defined in terms of an important lattice-isometric invariant of Nakano spaces, the essential range of the variable exponent.

1. Introduction

Douglas' Theorem ([Dou65]) states that if (Ω, Σ, μ) is a probability space, the conditional expectation operators are the only contractive projections on $L_1(\Omega, \Sigma, \mu)$ that leave constant functions invariant (in particular these projections are positive). As a consequence, the range of a contractive projection on $L_1(\Omega, \Sigma, \mu)$ can be represented as $L_1(\Omega_0, \Sigma_0, \mu_0)$ for some $\Omega_0 \subseteq \Omega$, $\Sigma_0 \subseteq \{A \cap \Omega_0 \mid A \in \Sigma\}$, with μ_0 denoting the restriction of μ to Σ_0. An elementary proof of Douglas' Theorem appears in [AAB93]. This result was extended to spaces $L_p(\Omega, \Sigma, \mu)$, for $1 < p < \infty$, by Andô ([And66]) for μ a probability measure, and to L_p-spaces of a general measure by Tzafriri ([Tza69]). For positive contractive projections, however, these structure results concerning the ranges become quite trivial, since the range of a positive contractive projection in an L_p-space is necessarily a closed sublattice, hence an L_p-space. Positive projections of a more general class of Banach function spaces have been characterized in terms of conditional expectation by Kulakova in [Kul81]. A complete description of order-continuous positive projections in ideals of measurable functions in terms of conditional expectation-type operators is given in [DHdP90]. A more recent survey article on contractive projections in Köthe function spaces and Banach sequence spaces is [Ran01].

The very general results of [DHdP90] do not necessarily give much insight on the structure of ranges of positive contractive projections in concrete Banach function spaces. As a simple example of such "concrete" spaces we consider here Nakano spaces. These spaces are generalizations of the Banach spaces $L_p(\mu)$ in which the exponent p is allowed to vary measurably over a set of values in $[1, \infty)$. They are also called "variable exponent Lebesgue spaces" in certain parts of the literature (the name that we adopt here is standard; see [FJ03, p. 127] or [JKL93]). In this paper, we show that in any nontrivial Nakano space $X = L_{p(\cdot)}(\Omega, \Sigma, \mu)$ with essentially bounded random exponent function $p(\cdot)$, the range $Y = R(P)$ of a positive contractive projection P is itself representable as a Nakano space $L_{p_Y(\cdot)}(\Omega_Y, \Sigma_Y, \nu_Y)$, for a certain measurable set $\Omega_Y \subseteq \Omega$ (the support of the range), a certain sub-sigma ring $\Sigma_Y \subseteq \Sigma$ (with

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maximal element Ω_Y) naturally determined by the lattice structure of Y, and a semi-finite measure ν_Y, namely the restriction of some measure ν on Σ which is equivalent to μ. Furthermore, we show that the random exponent $p_Y(\cdot)$ associated with such a range can be taken to be the restriction to Ω_Y of the random exponent $p(\cdot)$ (this restriction turns out to be Σ_Y-measurable).

An essential motivation for this study is to find a Banach lattice isometric characterization of the class of Nakano spaces or suitable subclasses. Such a characterization has been known since the 40's for L_p-spaces, and is due essentially to Bohnenblust (Kakutani for the $p = 1$ case). By the Kakutani-Bohnenblust theorem, for a given $p \in [1, \infty)$, the class of L_p-spaces (considered in the category of Banach lattices) is characterized by a single axiom:

$$\forall x \quad \|x\|^p = \|x_+\|^p + \|x_-\|^p$$

where $x_+ = x \vee 0$ and $x_- = x \wedge 0$ denote respectively the positive and negative parts of the element x in the Banach lattice. Like the previous one for L_p-spaces, the characterization we have in mind for the class of Nakano spaces should involve only the Banach lattice structure, and no extra features like the measure space (nor a disjoint additive modular in the spirit of [CZ78]); it should also be of local nature.

Let us first mention an important invariant in the order-isometric theory of Nakano spaces, which turns out to be the *essential range* of the exponent function $p(\cdot)$. A point q lies in this essential range if the measure $p(\mu)$ charges every interval in $\mathbb{R}$ centered on q. We call such an essential range (by abuse) the essential range of the Nakano space.

Then, as an application of the previous result on ranges of projections, for K a compact subset of $[1, \infty)$ we can provide a Banach lattice characterization of Nakano spaces with essential range included in K. This characterization is formulated in terms of pavings by finite-dimensional sublattices of a certain kind, and it resembles that of the Banach spaces $L_p(\mu)$ as those Banach spaces which can be paved by the spaces $(\ell_p^n \mid n \geq 1)$. We conclude with a similar characterization of Nakano spaces with essential range exactly equal to K.

Let K be a compact subset of $[1, \infty)$ and let $\mathcal{N}_K$ be the closure under lattice isometry of the class of all Nakano spaces $L_{p(\cdot)}(\Omega, \Sigma, \mu)$ for which the essential range of $p(\cdot)$ is exactly K. In [Poi06], the first author considered the question of whether the class $\mathcal{N}_K$ is closed under Banach-lattice ultraroots. The problem of characterizing the ranges of positive contractive projections in Nakano spaces was suggested by this question, since an ultraroot of a reflexive Nakano space is naturally the range of a positive contractive projection. The closure of $\mathcal{N}_K$ under ultraroots (Proposition 3.8.2 in [Poi06]) and the closure of the same class under ultraproducts (Corollary to Proposition 3.4.2 in [Poi06]) yield that the class $\mathcal{N}_K$ is axiomatizable in positive bounded logic for normed space structures (see [HI03]), or equivalently for continuous logic for metric structures (see [BYBHU]). In particular the results of section 3 and section 4 are essentially taken (in the smooth case) from the first author's thesis [Poi06] written under the direction of C. W. Henson. We refer the interested reader to this thesis for more results on the model theory of Nakano spaces, and also to the more recent paper [BY] by Ben Yaacov.

The paper is organized as follows: after a section of preliminaries about positive contractive projections, Musielak-Orlicz and Nakano spaces, we give in section 3 a description of norming functionals in smooth Nakano spaces, which will be the key for the structure theorem of the ranges of positive contractive projections of section 4. Sections 5 and 6 are devoted to the announced Banach lattice characterization of Nakano spaces with essential range included in (resp. equal to) a given compact set in $[1, \infty)$.

2. Preliminaries

If (Ω, Σ, μ) is a measure space, we denote by $L_0(\Omega, \Sigma, \mu)$ the space of equivalence classes of measurable real-valued functions defined on Ω, modulo equality μ-almost everywhere. If we regard it as equipped with the partial ordering $\leq$ defined by $[f] \leq [g] \iff \mu(\{\omega \in \Omega \mid f(\omega) > g(\omega)\}) = 0$, then $L_0(\Omega, \Sigma, \mu)$ is a topological Riesz space for the topology of convergence in measure. A *Köthe function space* over (Ω, Σ, μ) is a linear subspace and order ideal of $L_0(\Omega, \Sigma, \mu)$ that is order dense in $L_0(\Omega, \Sigma, \mu)$, equipped with a norm for which it is complete and which is compatible with the partial ordering on X. A Banach lattice X is said to be *order-continuous* if $\|x_\alpha\| \to 0$ for every downwards-directed net (x_α) of positive elements of X satisfying $\inf x_\alpha = 0$. In an order-continuous Köthe function space X the support of every element f is σ-finite. In what follows, we will restrict our attention to order-continuous Köthe function spaces.

A measure space (Ω, Σ, μ) is said to be *decomposable* if can be expressed as a disjoint union of measure spaces of finite measure, i.e. there exists a partition $\{\Omega_i \mid i \in I\}$ of Ω such that:

(i) $\Omega_i \in \Sigma$ and $\mu(\Omega_i) < \infty$ for all $i \in I$;
(ii) $\Sigma = \{A \subseteq \Omega \mid \Omega_i \cap A \in \Sigma \text{ for all } i \in I\}$;
(iii) For all $A \in \Sigma$, $\mu(A) = \Sigma_{i\in I}\mu(A \cap \Omega_i)$.

Every order-continuous Banach lattice can be represented as a Köthe function space over a decomposable measure space. Moreover, the dual of every Köthe function space can be represented as a Köthe function space over a decomposable measure space.

A Banach lattice X is said to be *strictly monotone* if whenever x and y are distinct elements of X satisfying $0 \leq x \leq y$, it is the case that $\|x\| < \|y\|$. If X is a strictly monotone Banach lattice and $P : X \to X$ is a positive contractive projection on X then $P(X)$ is a closed sublattice of X. (See [DHdP90, Lemma 4.5].)

Let X be a Köthe function space over a measure space (Ω, Σ, μ). Let f_0 be a positive element of $L_0(\Omega, \Sigma, \mu)$ (we will abbreviate this by writing $f_0 \in L_0(\Omega, \Sigma, \mu)_+$ in what follows), and let Σ_0 be a sub-σ-algebra of Σ. Following [HR07], we define

$$X_{f_0}(\Sigma_0) = \{h \in L_0(\Omega, \Sigma_0, \mu) \mid S(h) \subseteq S(f_0) \ \&\ f_0 \cdot h \in X\},$$

where $S(f)$ denotes the support of f. If X is a Köthe function space over a measure space (Ω, Σ, μ), $f_0 \in L_0(\Omega, \Sigma, \mu)_+$, and Σ_0 is a sub-σ-algebra of Σ, then $f_0 \cdot X_{f_0}(\Sigma_0)$ is a closed sublattice of X.

Let X be a Köthe function space over a measure space (Ω, Σ, μ), and let Y be a closed sublattice of X. We denote by Σ_Y^0 the set consisting of all the supports of elements of Y. By Σ_Y we mean the σ-algebra generated by Σ_Y^0.

Fact 2.1. *Let X be an order-continuous Köthe function space over a complete, decomposable measure space (Ω, Σ, μ). Then for every closed sublattice Y of X there exist a sub-σ-algebra Σ_Y of Σ and a positive Σ-measurable function f_0 such that $Y = f_0 \cdot X_{f_0}(\Sigma_Y)$. (See* [HR07]*.)*

Let X be an order-continuous Köthe function space over the measure space (Ω, Σ, μ) with a strictly monotone norm, and let Y be the range of a positive contractive projection P on X. Let Σ_P denote the closure of Σ_Y under taking suprema and infima of arbitrary families (with respect to inclusion) up to μ-null sets. Let S_Y denote the smallest element of Σ containing all the supports of elements in Y. (In particular, $S_Y \in \Sigma_P$.) The proof of the following useful criterion is contained in the paragraph immediately preceding Lemma 2.3 in [HR07].

Fact 2.2. *Let f be a Σ-measurable function. Then f is Σ_P-measurable if and only if its restriction to S_Y^c is constant and its restriction to any element of Σ_Y^0 is Σ_Y-measurable.*

As a consequence, if f is a Σ-measurable function with σ-finite support then f is Σ_P-measurable if and only if it is Σ_Y-measurable.

Let X be a Banach space. By S_X we denote the unit sphere of X, i.e. $S_X = \{x \in X \mid \|x\| = 1\}$. We define a *duality map* J from X into subsets of the dual Banach space X^* by the condition that $f \in J(x)$ if and only if $\|f\|_{X^*} = \|x\|_X$ and $\langle f, x\rangle = \|x\|_X^2$. If $J(x)$ contains exactly one functional then element x is called *smooth* in X, and we write $J : X \to X^*$ in this situation. If every element $x \in S_X$ is smooth in X then X is called *smooth*. Note that if there is a map $J : S_X \to S_{X^*}$ satisfying the condition that $y^* = J(x)$ if and only if $\langle y^*, x\rangle = 1$, then the map $\hat{J} : X \to X^*$ defined by $\hat{J}(x) = \|x\| J\left(\frac{x}{\|x\|}\right)$ if $x \neq 0$, $J(0) = 0$, is a duality map.

A Banach space is said to be *strictly convex* if its norm is *strictly convex*, i.e., if $\|\frac{x+y}{2}\| < \frac{\|x\|}{2} + \frac{\|y\|}{2}$ whenever x and y are distinct elements of X. If the conjugate space X^* is strictly convex (resp. smooth), then X is smooth (resp. strictly convex). (See [Die75, Chapter 2].) Note that a strictly convex Banach lattice is strictly monotone.

If a Banach lattice X is reflexive, it does not contain c_0 and hence it is order continuous. (See Propositions 1.a.5, 1.a.7 and 1.a.8 in [LT79].) Therefore, its Banach lattice dual X^* is itself a Köthe function space over the measure space (Ω, Σ, μ) (it in fact coincides with the Köthe dual X' of X).

Lemma 2.3. *Let X be a reflexive Köthe function space over the measure space (Ω, Σ, μ) with a smooth and strictly convex norm. Suppose P is a positive contractive projection on X. Then there exists a positive Σ-measurable function f_0' such that $P^*(X') = f_0' \cdot X'_{f_0'}(\Sigma_P)$, where P^* denotes the adjoint of P.*

Proof. This proof is an adaptation of the proof of Lemma 2.3 in [HR07]. If P^* denotes the adjoint of P, then P^* is a positive contraction. By the smoothness of the norm of X, the norm of X^* is strictly convex, hence strictly monotone. By the strict monotonicity of the norm of X^*, $P^*(X^*)$ is a closed sublattice of X^* and there exists a positive Σ-measurable function f_0' such that $P^*(X^*) = f_0' \cdot X_{f_0'}(\Sigma_{P^*(X^*)})$. The smoothness of X yields a single valued duality map $J : X \to X^*$. By a result of Calvert (see [Cal75]), $J(P(X)) \subseteq P^*(X^*)$. Since X^* is also smooth (X being strictly convex), J is a bijection from X onto X^* with inverse J^*, where $J^* : X^* \to X^{**} = X$ is the duality map of X^*. It results from the strict monotonicity of the norm of X that for every nonzero $h \in X$, we have that $S(Jh) = S(h)$, whence $\Sigma_{P^*(X^*)} = \Sigma_{P(X)}$. Now, X^* is order-continuous, and so its elements have σ-finite supports. By Fact 2.2, $X^*_{f_0'}(\Sigma_{P^*}) = X^*_{f_0'}(\Sigma_P)$. □

Musielak-Orlicz spaces. A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be an *Orlicz function* if $\varphi(0) = 0$, $\varphi(1) = 1$, and φ is convex. Given $k \geq 2$, if $\varphi(2t) \leq k\varphi(t)$ for all $t \in [0, \infty)$, we say that φ *satisfies the* Δ_2^k*-condition.* If φ satisfies the Δ_2^k-condition for some $k \geq 2$ we say that φ *satisfies the* Δ_2*-condition.*

Remark 2.4. In the literature, the requirement $\varphi(1) = 1$ above is often excluded from the definition of Orlicz function, and Orlicz functions φ satisfying such a requirement are said to be *normalized*. In this paper we will deal only with normalized Orlicz functions.

Let (Ω, Σ, μ) be a measure space. A *Musielak-Orlicz function* on (Ω, Σ, μ) is a function

$$\psi : [0, \infty) \times \Omega \to [0, \infty)$$

such that

A: $\psi(t, \cdot) \in L_0(\Omega, \Sigma, \mu)$ for every $t \in [0, \infty)$.
B: $\psi(\cdot, \omega)$ is an Orlicz function for all $\omega \in \Omega$.

Given a measure space (Ω, Σ, μ) and a Musielak-Orlicz function ψ on it, we define $\Psi : L_0(\Omega, \Sigma, \mu) \to [0, \infty]$ by

$$\Psi(f) = \int_\Omega \psi\left(|f(\omega)|, \omega\right) \, d\mu(\omega)$$

and set

$$\|f\|_\psi = \inf\left\{\alpha \in [0,\infty) \mid \Psi\left(\frac{f}{\alpha}\right) \leq 1\right\}.$$

Let

$$L_\psi(\Omega, \Sigma, \mu) = \left\{f \in L_0(\Omega, \Sigma, \mu) \mid \|f\|_\psi < \infty\right\}.$$

We say that such a space $L_\psi(\Omega, \Sigma, \mu)$ is a *Musielak-Orlicz space.* The functional Ψ is a convex modular on $L_\psi(\Omega, \Sigma, \mu)$. The norm $\|\cdot\|_\psi$ is usually called the *Luxemburg norm* in the literature. The *Orlicz norm* $\|\cdot\|_\psi^0$ on $L_\psi(\Omega, \Sigma, \mu)$ is defined by $\|f\|_\psi^0 = \inf\left\{\alpha \in [0,\infty) \,\middle|\, \frac{1+\Psi(\alpha f)}{\alpha} \leq 1\right\}$. The Luxemburg and Orlicz norms are equivalent and satisfy

$$\|f\|_\psi \leq \|f\|_\psi^0 \leq 2\|f\|_\psi$$

for $f \in L_\psi(\Omega, \Sigma, \mu)$. Orlicz norms will not be considered here, but they appear naturally on dual spaces of Musielak-Orlicz spaces.

Let $k \geq 2$. We say that a Musielak-Orlicz function ψ satisfies the Δ_2^k*-condition* if $\psi(\cdot, \omega)$ satisfies the Δ_2^k-condition for a.e. $\omega \in \Omega$. (The Musielak-Orlicz function ψ is said to satisfy the Δ_2*-condition* if it satisfies the Δ_2^k-condition for some $k \geq 2$.) In such a case, Ψ satisfies the Δ_2^k-condition as a convex modular. Recall that a Banach lattice $(X, \|\cdot\|)$ is said to be *order-continuous* if for every downwards directed net $(x_i \mid i \in I)$ of elements in X satisfying $\inf x_i = 0$ it is the case that $\|x_i\| \to 0$.

Let $L_\psi := L_\psi(\Omega, \Sigma, \mu)$ be a Musielak-Orlicz space. Then it contains a dense ideal H_ψ defined by

$$H_\psi = \{f \in L_0(\Omega, \Sigma, \mu) \mid \Psi(tf) < \infty \text{ for all } t \in (0, \infty)\}.$$

In case L_ψ satisfies the Δ_2^k-condition for some $k \geq 2$, then $H_\psi = L_\psi$ and L_ψ is order-continuous. (See pages 31–32 of [HLR91].)

A *convex modular* on a vector lattice E is a map $\Theta : E \to [0, \infty)$ satisfying:

$$\Theta(f) = 0 \iff f = 0;$$
$$|f| \leq |g| \implies \Theta(f) \leq \Theta(g);$$
$$\Theta(\alpha f + (1-\alpha)g) \leq \alpha\Theta(f) + (1-\alpha)\Theta(g);$$
$$|f| \wedge |g| = 0 \implies \Theta(f+g) = \Theta(f) + \Theta(g)$$

(for all $f, g \in E$ and $\alpha \in [0,1]$). Let E be a vector lattice and let Θ be a convex modular on E. We define $\|\cdot\|_\Theta : E \to [0, \infty]$ by

$$\|f\|_\Theta = \inf\{t \in (0,\infty) \mid \Theta(f/t) \leq 1\}$$

for all $f \in E$. If E is a vector lattice and Θ a convex modular on E, then $(E, \|\cdot\|_\Theta)$ is a normed vector lattice. (See, for example, [Mus83, page 7].) If E is a Banach lattice E equipped with a convex modular Θ, then (E, Θ) is said to be an *Orlicz lattice* if it the norm on E coincides with the norm induced by Θ on E.

The following representation theorem appears in [HLR91] as Theorem 3.17. We will only need to apply it in cases in which the Δ_2^k-condition is satisfied for some $k \geq 2$, in which case $H_\psi = L_\psi$.

Theorem 2.5. *Let (E, Θ) be an Orlicz lattice. Then there exist a measure space (Ω, Σ, μ) and a Musielak-Orlicz function ψ on Ω such that (E, Θ) is isomodularly isomorphic to $(H_\psi(\Omega, \Sigma, \mu), \Psi)$ as Orlicz lattices.*

Remark 2.6. In the formulation of this result appearing in [HLR91], it is remarked that the representation can be achieved with the additional requirement that the Musielak-Orlicz function ψ be such that $\psi(\cdot,\omega)$ belongs to the closure (in the topology of pointwise convergence) of the convex hull of the set D of all functions $t \mapsto \frac{\Theta(tx)}{\Theta(x)}$, where $x \in E$.

Let (Ω,Σ,μ) be a measure space. Let $p(\cdot)$ denote an element of $L_0(\Omega,\Sigma,\mu)$ satisfying $\operatorname{ess\,inf} p(\cdot) \geq 1$. Define a Musielak-Orlicz function $\psi : [0,\infty) \times \Omega \to [0,\infty)$ by writing $\psi(t,\omega) = t^{p(\omega)}$. The Musielak-Orlicz space $L_\psi(\Omega,\Sigma,\mu)$ thus obtained is denoted by $L_{p(\cdot)}(\Omega,\Sigma,\mu)$ and is referred to as a *Nakano space* with random exponent $p(\cdot)$. Its convex modular is denoted by $\Theta_{p(\cdot)}$. In this paper, we will be interested in Nakano spaces satisfying the condition that $\operatorname{ess\,sup} p(\cdot) < \infty$, and so we adopt the convention that all Nakano spaces satisfy this condition. Note that Nakano spaces are Köthe function spaces that generalize the classical Banach lattices $L_p(\mu)$.

Remark 2.7. A Nakano space satisfies the Δ_2-condition (and is thus order-continuous as a Banach lattice) if and only if its random exponent is essentially bounded.

For $p(\cdot) \in L_0(\Omega,\Sigma,\mu)$ given, we define its *essential range* $\mathcal{R}_{p(\cdot)}$ to be the set of all $q \in [1,\infty)$ such that for all $\epsilon > 0$, the set $\{\omega \in \Omega \mid |p(\omega) - q| < \epsilon\}$ has positive μ-measure.

Remark 2.8. Proposition 2.11 below shows that the essential range $K := \mathcal{R}_{p(\cdot)}$ of the random exponent of $L_{p(\cdot)}(\Omega,\Sigma,\mu)$ is an invariant under order-isometries. This justifies saying that K is *the essential range* associated to any Nakano space X which is isometric to $L_{p(\cdot)}(\Omega,\Sigma,\mu)$. We will often say that X is a Nakano space with essential range K to mean that K is the essential range associated to the Nakano space X; or, more precisely, that K is the essential range of some random exponent $p(\cdot)$ for which the Nakano space $L_{p(\cdot)}(\Omega,\Sigma,\mu)$ is isometric to X.

Note that the essential range associate to a Nakano space is necessarily a closed subset of $[1,\infty)$, and it is in fact compact whenever the random exponent $p(\cdot)$ of the Nakano space is essentially bounded.

Fact 2.9. *A sufficient condition for a Nakano space X to be reflexive is that it has associated compact essential range $K \subseteq (1,\infty)$.*

This follows from the more general fact that a Musielak-Orlicz space L_ψ is reflexive whenever both ψ and its conjugate ψ^* satisfy the Δ_2-condition. (See e.g. [Mus83, p. 189].)

Conversely, if X is a reflexive Nakano space over a non-atomic measure space, then its associated essential range K must satisfy $K \subseteq (1,\infty)$ (this is Corollary 2.7 in [KR91]), but this is no longer true if the measure space has atoms.

It is useful to note that whenever $p(\cdot) > 1$ a.e. and $\operatorname{ess\,sup} p(\cdot) < \infty$, the Nakano space $L_{p(\cdot)}(\Omega,\Sigma,\mu)$ is strictly convex.

In section 4 of this paper we will make use of the classical procedure of r-convexification to pass from nonsmooth Nakano spaces to smooth ones. For a general discussion of these procedures, the reader is referred to [LT79]. If X is a Köthe function space over (Ω,Σ,μ) and $r \geq 1$, then the *r-convexification* of X is the Köthe function space $X^{(r)} = \{f \in L_0(\Omega,\Sigma,\mu) \mid |f|^r \in X\}$ equipped with the norm

$$\|f\|_{X^{(r)}} = \||f|^r\|_X^{\frac{1}{r}}.$$

Let K be a compact subset of $[1,\infty)$. If X is a Nakano space with essential range $= K$ and $r \geq 1$, then $X^{(r)}$ is a Nakano space with associated essential range rK.

Essential range invariance under isometries. In this brief subsection we provide a proof that for Nakano spaces of dimension ≥ 2 with essentially bounded exponent function, the associated essential range remains invariant under lattice isometry.

Lemma 2.10. *Let $X := L_p(\cdot)(\Omega, \Sigma, \mu)$ be a Nakano space and set $p_* = \operatorname{ess\,inf}\, p(\cdot)$ and $p^* = \operatorname{ess\,sup}\, p(\cdot)$. Then X is p_*-convex and p^*-concave (both with constant 1).*

Proof. Since $p(\cdot)/p^* \leq 1$ a. e., the map $t \mapsto t^{p(\omega)/p^*}$ is concave for a.e. ω. Hence the "modular" $\Theta_{p(\cdot)/p^*}$ is concave on $L_0^+(\Omega, \Sigma, \mu)$. Let $x_1, ..., x_n \in X$ and set $\alpha = (\sum_{i=1}^n \|x_i\|^{p^*})^{1/p^*}$ and $x = (\sum_{i=1}^n |x_i|^{p^*})^{1/p^*}$. Then

$$\begin{aligned}\Theta_{p(\cdot)}\left(\frac{x}{\alpha}\right) &= \Theta_{\frac{p(\cdot)}{p^*}}\left(\frac{x^{p^*}}{\alpha^{p^*}}\right) = \Theta_{\frac{p(\cdot)}{p^*}}\left(\sum_{i=1}^n \frac{\|x_i\|^{p^*}}{\alpha^{p^*}} \cdot \frac{|x_i|^{p^*}}{\|x_i\|^{p^*}}\right) \\ &\geq \sum_{i=1}^n \frac{\|x_i\|^{p^*}}{\alpha^{p^*}} \Theta_{\frac{p(\cdot)}{p^*}}\left(\frac{|x_i|^{p^*}}{\|x_i\|^{p^*}}\right) = \sum_{i=1}^n \frac{\|x_i\|^{p^*}}{\alpha^{p^*}} \Theta_{p(\cdot)}\left(\frac{x_i}{\|x_i\|}\right) = 1.\end{aligned}$$

Hence $\|x\| = \left\|(\sum_{i=1}^n |x_i|^{p^*})^{1/p^*}\right\| \geq \alpha = (\sum_{i=1}^n \|x_i\|^{p^*})^{1/p^*}$, which proves that X is p^*-concave with constant 1. The proof of the p_*-convexity is similar (it can also be observed that X is the p_*-convexification of the convex Nakano space $Y := L_{p(\cdot)/p_*}(\Omega, \Sigma, \mu)$). ☐

Proposition 2.11. *Fix $M \geq 1$. Let $L_{p_1(\cdot)} := L_{p_1(\cdot)}(\Omega_1, \Sigma_1, \mu_1)$ be a Nakano space of dimension ≥ 2 with essential range included in $[1, M]$. Let $L_{p_2(\cdot)} := L_{p_2(\cdot)}(\Omega_2, \Sigma_2, \mu_2)$ be another Nakano space which is lattice-isometrically isomorphic to $L_{p_1(\cdot)}$. Then $\mathcal{R}_{p_2(\cdot)} = \mathcal{R}_{p_1(\cdot)}$.*

Proof. Let us show first that if $L_{p_1(\cdot)}$ is lattice isomorphic to $L_{p_2(\cdot)}$ then $(p_1)^* = \operatorname{ess\,sup}\, p_1(\cdot)$ and $(p_2)^* = \operatorname{ess\,sup}\, p_1(\cdot)$ are equal. Assume for example that $(p_1)^* < (p_2)^*$. Since we know by the preceding Lemma that $L_{p_1(\cdot)}$ is $(p_1)^*$-concave with constant 1, it is sufficient to prove that $L_{p_2(\cdot)}$ is not for reaching a contradiction. Observe that for every $\epsilon > 0$, $A = p_2(\cdot)^{-1}([(p_2)^* - \epsilon, (p_2)^*])$ verifies $\mu_2(A) > 0$. Choose $\epsilon \in (0, (p_2)^* - (p_1)^*)$. Let $y \in \chi_A L_{p_2(\cdot)}$ and $x \in L_{p_2(\cdot)}$ be two disjointly supported positive elements of norm 1. Then for every $\lambda \in (0, 1)$:

$$\Theta_{p_2(\cdot)}(x + \lambda y) = \Theta_{p_2(\cdot)}(x) + \Theta_{p_2(\cdot)}(\lambda y) \leq \Theta_{p_2(\cdot)}(x) + \lambda^{(p_2)^* - \epsilon}\Theta_{p_2(\cdot)}(y) = 1 + \lambda^{(p_2)^* - \epsilon}.$$

Hence $\|x + \lambda y\| \leq 1 + \lambda^{(p_2)^* - \epsilon}$. On the other hand

$$(\|x\|^{(p_1)^*} + \|\lambda y\|^{(p_1)^*})^{1/(p_1)^*} = (1 + \lambda^{(p_1)^*})^{1/(p_1)^*} = 1 + \frac{1}{(p_1)^*}\lambda^{(p_1)^*} + o(\lambda^{(p_1)^*}).$$

Since $\lambda^{(p_1)^*} \gg \lambda^{(p_2)^* - \epsilon}$ when $\lambda \to 0$, we see that $L_{p_2(\cdot)}$ is not $(p_1)^*$-concave with constant 1.

Let $(p_1)_* = \operatorname{ess\,inf}\, p_1(\cdot)$ and $(p_2)_* = \operatorname{ess\,inf}\, p_2(\cdot)$, and assume that $(p_1)_* \neq (p_2)_*$. Without loss of generality, assume further that $(p_2)_* + \epsilon < (p_1)_*$ for some $\epsilon > 0$. Observe that $B = p_2(\cdot)^{-1}([(p_2)_*, (p_2)_* + \epsilon])$ verifies $\mu_2(B) > 0$. Let $y \in \chi_B L_{p_2(\cdot)}$ and $x \in L_{p_2(\cdot)}$ be two disjointly supported positive elements of norm 1. Then we have that

$$\begin{aligned}\Theta_{p_2(\cdot)}\left(\frac{x + \lambda y}{(1 + \lambda^{(p_2)_* + \epsilon})^{1/(p_2)^*}}\right) &\geq \frac{1}{1 + \lambda^{(p_2)_* + \epsilon}}\Theta_{p_2(\cdot)}(x + \lambda y) \\ &\geq \frac{1}{1 + \lambda^{(p_2)_* + \epsilon}}\left(\Theta_{p_2}(x) + \lambda^{(p_2)_* + \epsilon}\Theta_{p_2(\cdot)}(y)\right) = 1.\end{aligned}$$

Since $\lambda^{(p_1)_*} \ll \lambda^{(p_2)_* + \epsilon}$, we have that

$$\|x + \lambda y\| \geq (1 + \lambda^{(p_2)_* + \epsilon})^{1/(p_2)^*} = 1 + \frac{1}{(p_2)^*}\lambda^{(p_2)_* + \epsilon} + o\left(\lambda^{(p_2)_* + \epsilon}\right) \gg 1 + \frac{1}{(p_1)_*}\lambda^{(p_1)_*} + o\left(\lambda^{(p_1)_*}\right),$$

whence $\|x+\lambda y\| \nleq \left(\|x\|^{(p_1)_*} + \|\lambda y\|^{(p_1)_*}\right)^{1/(p_1)_*}$. This, in turn, implies that $L_{p_1(\cdot)}$ is not $(p_1)_*$-convex with constant 1, a contradiction. Therefore $(p_1)_* = (p_2)_*$.

Let T be an isometric lattice isomorphism from $L_{p_2(\cdot)}$ onto $L_{p_1(\cdot)}$. Let $K_1 = \mathcal{R}_{p_1(\cdot)}$ and $K_2 = \mathcal{R}_{p_2(\cdot)}$. If $L_{p_1(\cdot)}$ and $L_{p_2(\cdot)}$ are 2-dimensional, it results from the preceding that $K_1 = K_2$. So let us suppose that $L_{p_1(\cdot)}$ and $L_{p_2(\cdot)}$ have dimension ≥ 3. Let $p \in K_1$, $A_1 = p_1^{-1}([1,p])$ and $A_2 = p_1^{-1}((p,M])$. For $i = 1,2$, let $Y_i = \chi_{A_i} L_{p_1(\cdot)}$ and $Z_i = T(Y_i)$. Then Z_i is a band in $L_{p_2(\cdot)}$, say with support B_i, for $i = 1,2$. Note that at least one of Y_1, Y_2 must have dimension ≥ 2, since otherwise $L_{p_1(\cdot)} = Y_1 \oplus Y_2$ would have dimension < 3. If $\dim(Y_1) > 1$ then we apply the previous results to prove that $(p_2 \restriction_{B_1})^* = (p_1 \restriction_{A_1})^*$. Similarly, if $\dim(Y_2) > 1$ we have $(p_2 \restriction_{B_2})_* = (p_1 \restriction_{A_2})_*$. In any case, $p \in K_2$, and so $K_1 \subseteq K_2$. The proof that $K_2 \subseteq K_1$ is similar. □

3. Duality mapping in smooth Nakano spaces

A functional F on a Banach space X is said to be *Gâteaux-differentiable* at the point $f \in X$ if, for arbitrary $h \in X$, the function $F(f+th)$ is differentiable with respect to t and the derivative of this function, for $t = 0$, has the form

$$\frac{d}{dt}F(f+th)\bigg|_{t=0} = \langle g, h\rangle,$$

where the element g from the topological dual X^* to the space X does not depend on h, and $\langle\cdot,\cdot\rangle$ denotes the usual duality pairing. The element g will be called the *Gâteaux gradient* of the functional F at the point f. The map $\nabla F : X \to X^*$, defined by the formula $\nabla F(f) = g$ on all elements f at which F is Gâteaux differentiable, will also be called the Gâteaux gradient.

Fact 3.1. *Let X be a Banach space and let $x \in X \setminus \{0\}$. Then x is a smooth point if and only if the norm is Gâteaux differentiable at the point x. The unique value at x of the duality map J is then $J(x) = \|x\|g$, where g is the Gâteaux gradient of the norm at x.*

Given $p(\cdot) \in L_0(\Omega,\Sigma,\mu)$ with $p(\cdot) > 1$ a.e., note that the function $\psi(\cdot,\omega) : \mathbb{R}\times\Omega \to [0,\infty)$ defined by $\psi(t,\omega) = |t|^{p(\omega)}$ is continuously differentiable at every point $t \in \mathbb{R}$, for every $\omega \in \Omega$. Let $f, g \in L_{p(\cdot)}(\Omega,\Sigma,\mu)$. For $r, h \in \mathbb{R}$, $s > 0$, and $\omega \in \Omega$, define $\xi(r,s,\omega) = \psi\left(\frac{f(\omega)+rg(\omega)}{s},\omega\right)$ and $\Delta\xi(r,h,s,\omega) = \frac{\xi(r+h,s,\omega)-\xi(r,s,\omega)}{h}$. Then

$$\lim_{h\to 0}\Delta\xi(r,h,s,\omega) = \frac{1}{s}\frac{\partial\psi}{\partial t}\left(\frac{f(\omega)+rg(\omega)}{s},\omega\right)g(\omega)$$

for every $\omega \in \Omega$.

Note that $\int_\Omega \xi(r,s,\omega)\,d\mu(\omega)$ and $\int_\Omega \xi(r\pm 1,s,\omega)\,d\mu(\omega)$ all exist, whence so does

$$\int_\Omega \Delta\xi(r,\pm 1,s,\omega)\,d\mu(\omega) = \int_\Omega \xi(r\pm 1,s,\omega)\,d\mu(\omega) - \int_\Omega \xi(r,s,\omega)\,d\mu(\omega).$$

By the convexity of $\psi(\cdot,\omega)$, we have that $\Delta\xi(r,-1,s,\omega) \leq \Delta\xi(r,h,s,\omega) \leq \Delta\xi(r,1,s,\omega)$ for $|h| \leq 1$ and for a.e. $\omega \in \Omega$. Hence

$$|\Delta\xi(r,h,s,\omega)| \leq \max\left[|\Delta\xi(r,-1,s,\omega)|, |\Delta\xi(r,1,s,\omega)|\right].$$

The right hand side of the previous inequality is a function in L_1 not depending on h, and so we may apply Lebesgue's Dominated Convergence Theorem to obtain

$$\lim_{h\to 0}\int_\Omega \Delta\xi(r,h,s,\omega)\,d\mu(\omega) = \frac{1}{s}\int_\Omega \frac{\partial\psi}{\partial t}\left(\frac{f(\omega)+rg(\omega)}{s},\omega\right)g(\omega)\,d\mu(\omega).$$

Define Φ by

$$\Phi(r,s) = \int_\Omega \psi\left(\frac{f(\omega)+rg(\omega)}{s},\omega\right)\,d\mu(\omega).$$

The discussion above shows that $\frac{\partial\Phi}{\partial r}(r,s)$ exists. In fact,

$$\frac{\partial\Phi}{\partial r}(r,s) = \frac{1}{s}\int_\Omega \frac{\partial\psi}{\partial t}\left(\frac{f(\omega)+rg(\omega)}{s},\omega\right) g(\omega)\,d\mu(\omega).$$

By a similar argument we can show that

$$\frac{\partial\Phi}{\partial s}(r,s) = -\frac{1}{s^2}\int_\Omega \frac{\partial\psi}{\partial t}\left(\frac{f(\omega)+rg(\omega)}{s},\omega\right)(f(\omega)+rg(\omega))\,d\mu(\omega).$$

Moreover, both $\frac{\partial\Phi}{\partial r}(r,s)$ and $\frac{\partial\Phi}{\partial s}(r,s)$ are continuous as functions of (r,s) for fixed f,g. Since $\psi(t,\omega) = |t|^{p(\cdot)}$, we have that $\int_\Omega \psi\left(\frac{f(\omega)}{k},\omega\right)\,d\mu(\omega) = 1$ implies that $\|f\|_{p(\cdot)} = k$.

Remark 3.2. The norm $\|\cdot\|_{p(\cdot)}$ can be defined with the aid of the equation $\int_\Omega \psi\left(\frac{f(\omega)}{k},\omega\right)\,d\mu(\omega) = 1$ whenever $\|f\|_{p(\cdot)} \neq 0$.

Theorem 3.3. *If $p(\cdot) > 1$ a.e., and $\operatorname{ess\,sup} p(\cdot) < \infty$, then the norm $\|\cdot\|_{p(\cdot)}$ is a Gâteaux differentiable functional, and the duality map*

$$J : L_{p(\cdot)}(\Omega,\Sigma,\mu) \to \left(L_{p(\cdot)}(\Omega,\Sigma,\mu)\right)^*$$

is defined by

$$J(f)(g) = \left\langle \frac{1}{A}\cdot\frac{p(\omega)\ \operatorname{sgn}(f(\omega))\,|f(\omega)|^{p(\omega)-1}}{\|f\|_{p(\cdot)}^{p(\omega)-2}}, g(\omega)\right\rangle,$$

where the constant A is defined by

$$A := \int_\Omega \frac{p(\omega)|f(\omega)|^{p(\omega)}}{\|f\|_{p(\cdot)}^{p(\omega)}}\,d\mu(\omega).$$

Remark 3.4. Let $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. The conjugate space $(L_{p(\cdot)}(\Omega,\Sigma,\mu))^*$ may be represented as the Nakano space $L_{p'(\cdot)}(\Omega,\Sigma,\mu)$ equipped with the Orlicz norm.

Proof. Consider the equation

$$\int_\Omega \psi\left(\frac{f(\omega)+rg(\omega)}{s},\omega\right)\,d\mu(\omega) = 1.$$

This equation defines r as an implicit function of s. Since the partial derivatives $\frac{\partial\Phi(r,s)}{\partial r}$ and $\frac{\partial\Phi(r,s)}{\partial s}$ are continuous and since $\frac{\partial\Phi(0,s)}{\partial s} < 0$ whenever $\|f\| \neq 0$, by the Implicit Function Theorem we obtain that $\frac{\partial s}{\partial r}\Big|_{r=0} = \langle h, g\rangle$, where

$$h = \frac{\frac{\partial\psi}{\partial t}\left(\frac{f(\omega)}{\|f\|_{p(\cdot)}},\omega\right)}{\displaystyle\int_\Omega \frac{\partial\psi}{\partial t}\left(\frac{f(\omega)}{\|f\|_{p(\cdot)}},\omega\right)\frac{f(\omega)}{\|f\|_{p(\cdot)}}\ d\mu(\omega)}.$$

This completes the proof. ∎

Remark 3.5. The formula above for the duality mapping J is a special case of a similar formula in the setting of smooth Musielak-Orlicz spaces. See, for example, [JKL93].

As a quick consequence we have the following

Corollary 3.6. *Let $p(\cdot) \in L_0(\Omega, \Sigma, \mu)$ with $p(\cdot) > 1$ a.e. and $\operatorname{ess\,sup} p(\cdot) < \infty$. Then for any $f \in L_{p(\cdot)}(\Omega, \Sigma, \mu)$ with $\|f\|_{p(\cdot)} = 1$ we have that*

$$J(f) = \frac{p(\omega)\ \operatorname{sgn}(f(\omega))\, f(\omega)^{p(\omega)-1}}{\displaystyle\int_\Omega p(\omega)|f(\omega)|^{p(\omega)}\, d\mu(\omega)}.$$

We finish this section by providing a necessary and sufficient condition for the Gâteaux differentiability of the norm of a Nakano space.

Corollary 3.7. *Let $L_{p(\cdot)}(\Omega, \Sigma, \mu)$ be a Nakano space of dimension ≥ 2. The following conditions are equivalent:*

(i) $\|\cdot\|_{p(\cdot)}$ *is Gâteaux differentiable.*
(ii) $p(\cdot) > 1$ *a.e. and* $\operatorname{ess\,sup} p(\cdot) < \infty$.

Proof. ($\Leftarrow$) is Theorem 3.3.

($\Rightarrow$) If $\operatorname{ess\,sup} p(\cdot) = \infty$, it results from known facts about Musielak-Orlicz spaces that l_∞ embeds isometrically in $L_{p(\cdot)}$ (see [Hud84] in the non atomic case and [Kam82] in the purely atomic case). Since the norm of l_∞ is not smooth, neither is that of $L_{p(\cdot)}$.

If $p(\cdot) = 1$ on a set of positive measure, then we can find $A \in \Sigma$ with $0 < \mu(A) < \mu(\Omega)$, $\mu(\Omega \setminus A) > 0$, and $p(\cdot) \restriction_A \equiv 1$. Let $g = \frac{\chi_A}{\|\chi_A\|_{p(\cdot)}}$ and let $f \in L_{p(\cdot)}$ satisfy $f \wedge g = 0$ and $\|f\|_{p(\cdot)} = 1$. We may suppose that $\operatorname{ess\,sup} p(\cdot) \leq M < \infty$ on the support of f, for some $M > 0$. Define $\Phi : \mathbb{R} \times [0, \infty) \to [0, \infty)$ by

$$\Phi(r, s) := \Theta_{p(\cdot)}\left(\frac{f + rg}{s}\right) = \Theta_{p(\cdot)}\left(\frac{f}{s}\right) + \frac{|r|}{s}.$$

We have that

$$\begin{aligned} s = \|f + rg\|_{p(\cdot)} &\iff \Theta_{p(\cdot)}\left(\frac{f}{s}\right) + \frac{|r|}{s} = 1 \\ &\iff |r| = s - s\Theta_{p(\cdot)}\left(\frac{f}{s}\right). \end{aligned}$$

Let $\rho(s) = s - s\Theta_{p(\cdot)}\left(\frac{f}{s}\right)$ for $s \in (0, \infty)$. The function $\rho : (0, \infty) \to \mathbb{R}$ has derivative ρ' defined by

$$\rho'(s) = 1 + \int_\Omega (p(\omega) - 1)\left|\frac{f(\omega)}{s}\right|^{p(\omega)} d\mu(\omega).$$

Note that ρ' is strictly positive (in fact $\rho'(s) \geq 1$ for all $s \in (0, \infty)$). Let σ denote the inverse function of ρ. Then $\sigma : \mathbb{R} \to (0, \infty)$ is differentiable with $0 < \sigma' < \infty$ everywhere and satisfies $\sigma(0) = 1$ and

$$\sigma'(0) = \frac{1}{\rho'(1)} = \frac{1}{\int_\Omega p(\omega)|f(\omega)|^{p(\omega)}\, d\mu(\omega)}.$$

If we set $s(r) = \|f + rg\|_{p(\cdot)}$ we have $s(r) = \sigma(|r|)$; hence $s'_+(0) = \sigma'(0)$ and $s'_-(0) = -\sigma'(0)$, and since $\sigma'(0) > 0$, the norm $\|\cdot\|_{p(\cdot)}$ is not differentiable at f in the direction g. □

4. The range of a positive contractive projection in Nakano spaces

Let P be a positive contractive projection in a reflexive Nakano space $X = L_{p(\cdot)}(\Omega, \Sigma, \mu)$ onto a closed sublattice Y, and let J denote the duality mapping of X. If P^* denotes the adjoint of P, then Calvert's result in [Cal75] ensures that

$$J(Y) \subseteq P^* \left((L_{p(\cdot)}(\Omega, \Sigma, \mu))^*\right).$$

Theorem 4.1. *Suppose* $\operatorname{ess\,sup} p(\cdot) < \infty$. *Whenever the range of a positive contractive projection P on the Nakano space $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ is not 1-dimensional, then it is of the form $f_0 \cdot X_{f_0}(\Sigma_P)$ for some Σ-measurable f_0, with $p(\cdot) \restriction_{S(f_0)}$ being Σ_P-measurable.*

As an immediate consequence of this we have the following

Corollary 4.2. *Let $X := L_{p(\cdot)}(\Omega, \Sigma, \mu)$ be a Nakano space with compact essential range $K \subseteq [1, \infty)$. Then the range $R(P)$ of a positive contractive projection P on X satisfies $R(P) = f_0 \cdot L_{p(\cdot)\restriction_{\Omega_0}}(\Omega_0, \Sigma_0, \mu_0)$, where $\Omega_0 \in \Sigma_P$, $\Sigma_0 = \{A \cap \Omega_0 \mid A \in \Sigma_P\}$, and $\mu_0 = (f_0^{p(\cdot)}\mu) \restriction_{\Sigma_0}$.*

Proof of Theorem 4.1. We consider first the case in which $K \subseteq (1, \infty)$. In this case, X is reflexive, the norm of X is strictly convex and smooth, the range of P is a closed sublattice of X of the form $Y := f_0 \cdot X_{f_0}(\Sigma_P)$, for some $f_0 \in L_0(\Omega, \Sigma, \mu)$, and the range of P^* is $R(P^*) = f_0' \cdot X'_{f_0'}(\Sigma_P)$ (by Fact 2.1 and Lemma 2.3). It remains to show that $p(\cdot) \restriction S(f_0)$ is Σ_P-measurable. In order to establish this, it is sufficient to show that the restriction of $p(\cdot)$ to the support of any positive element in Y is Σ_Y-measurable. In fact, since Y is not 1-dimensional, we may and do reduce to considering elements in Y that are not atoms.

Let f be a positive element of Y that is not an atom.

Claim. There exist positive $g, h \in S_Y$ with $S(g) = S(h) = S(f)$ such that

$$\mu\left(\{\omega \in \Omega \mid g(\omega) = h(\omega)\}\right) = 0.$$

Proof. Write $f = f_1 + f_2$ for $f_1 \wedge f_2 = 0$, and let $g = \frac{f}{\|f\|}$. Let $h_0 = f_1 + 2f_2$ and $h = \frac{h_0}{\|h_0\|}$. Then we have that $\|f\| < \|h_0\| < 2\|f\|$. This yields the desired conclusion. □

So we have $g, h \in Y$ satisfying $S(g) = S(h)$. Then both $\frac{g}{h}$ and $\frac{J(g)}{J(h)}$ are Σ_P-measurable. We have that

$$\left|\frac{J(g)}{J(h)}\right| = \left|\frac{g}{h}\right|^{p(\cdot)\restriction_{S(g)}-1} \left|\frac{\displaystyle\int_\Omega p(\omega)|h(\omega)|^{p(\omega)} d\mu(\omega)}{\displaystyle\int_\Omega p(\omega)|g(\omega)|^{p(\omega)}\, d\mu(\omega)}\right|,$$

and so

$$\log\left|\frac{J(g)}{J(h)}\right| = \left(p(\cdot) \restriction_{S(g)} -1\right) \log\left|\frac{g}{h}\right| + \log\left|\frac{\displaystyle\int_\Omega p(\omega)|h(\omega)|^{p(\omega)} d\mu(\omega)}{\displaystyle\int_\Omega p(\omega)|g(\omega)|^{p(\omega)}\, d\mu(\omega)}\right|.$$

This readily shows that $p(\omega) \cdot \chi_{S(f)}$ is Σ_P-measurable, as everything else occurring in the above equation is Σ_P-measurable. Because f is arbitrary, $p(\cdot) \restriction_{S(f_0)}$ is Σ_P-measurable. This concludes the proof for the smooth case.

Let us now consider the general case. We know that the range of P is of the form $R(P) = f_0 \cdot X_{f_0}(\Sigma_P)$, for some Σ-measurable nonnegative f_0, and Σ_P is a sub-σ-algebra of Σ. Moreover the support of f_0 coincides with the support S_P of $R(P)$, that is the supremum of the supports of the elements of $R(P)$.

We may assume that f_0 is the indicator function of the Σ-measurable set S_P. (Simply define a new measure ν on Σ_P by requiring $|f_0|^{p(\cdot)}$ to be its Radon-Nikodym derivative with respect to μ.) In this case, the set Ω_P coincides with the supremum of the family of Σ_P-measurable sets A such that χ_A belongs to $R(P)$: such sets are clearly of finite ν-measure (in other words the measure ν_P induced by ν on $\Sigma'_P = \{A \cap S_P \mid A \in \Sigma_P\}$ is semi-finite).

A function $\phi : S_P \to \mathbb{R}$ is Σ'_P-measurable if and only if all of its restrictions to the support A of an element of $R(P)$ is Σ'_P-measurable; by the preceding it is sufficient to test this measurability for the restriction of ϕ to ν-finite $A \in \Sigma'_P$. We want to do that for $\phi = p(\cdot)$.

Fix now $A_0 \in \Sigma'_P$ with $\nu(A_0) < \infty$. Since P is positive, for every Σ-measurable function f with $|f| \leq c\chi_{A_0}$ for some $c \geq 0$, we have $|Pf| \leq c\chi_{A_0}$. Consequently P restricts to a map $P_{A_0} : \chi_{A_0}X \to \chi_{A_0}X$ which induces a map (still denoted by P_{A_0}) $L_\infty(A_0, \Sigma_{A_0}, \nu_{A_0}) \to L_\infty(A_0, \Sigma_{A_0}, \nu_{A_0})$. Note that $\|P_{A_0}f\|_\infty \leq \|f\|_\infty$, so P_{A_0} is a contractive linear projection of $L_\infty(A_0, \Sigma_{A_0}, \nu_{A_0})$.

Let r be any real number with $r > 1$, and X^r be the r-convexification of X. Note that $\chi_{A_0}X^r \subset \chi_{A_0}X$: indeed, if $f \in \chi_{A_0}X^r$ let $B_f = \{\omega \mid |f(\omega)| > 1\}$ and $C_f = A_0 \setminus B_f$, then $\chi_{C_f}f \in L_\infty(A_0) \subset \chi_{A_0}X$ while $\chi_{B_f}|f| \leq \chi_{B_f}|f|^r \in \chi_{A_0}X$, so $\chi_{B_f}f \in \chi_{A_0}X$, and finally $f = \chi_{B_f}f + \chi_{C_f}f \in \chi_{A_0}X$.

It is a standard fact that P, being a positive linear contraction both in $\chi_{A_0}X$ and in $L_\infty(A_0)$, induces a contraction in $\chi_{A_0}X^r$ [KPS82, page 246]. Here is an elementary proof of this fact. Let $f \in \chi_{A_0}X^r$ be non negative, set $g = f^r$ which belongs to X. Let $\theta = 1/r$. By the comparison between the arithmetic and geometric means we have that

$$f = g^\theta \chi_{A_0}^{1-\theta} \leq \theta g + (1-\theta)\chi_{A_0},$$

and hence

$$Pf \leq \theta Pg + (1-\theta)P\chi_{A_0} = \theta Pf^r + (1-\theta)\chi_{A_0}.$$

Applying this result to $f' = t^{-\theta}f$, for some real number $t > 0$, we obtain

$$Pf \leq \theta t^{\theta-1}Pf^r + (1-\theta)t^\theta\chi_{A_0}.$$

Since this is valid for every $t > 0$ we have that

$$Pf \leq \bigwedge_{t>0}(\theta t^{\theta-1}Pf^r + (1-\theta)t^\theta\chi_{A_0}) = (Pf^r)^\theta\chi_{A_0}^{1-\theta} = (Pf^r)^{1/r}$$

(for determining the infimum, note that we can restrict to rational values of t and then calculate pointwise). Thus $(Pf)^r \leq Pf^r \in X$, and so $Pf \in \chi_{A_0}X^r$ and $\|Pf\|_{X^r}^r = \|(Pf)^r\|_X \leq \|Pf^r\|_X \leq \|f^r\|_X = \|f\|_{X^r}^r$.

Now, $\chi_{A_0}X^r$ is a Nakano space with exponent function $rp(\cdot)$ (hence smooth) on which P acts as a positive linear contraction. Hence $rp(\cdot)_{|A_0}$ is Σ_P-measurable, and so is $p(\cdot)_{|A_0}$. Since A_0 is arbitrary, $p(\cdot)$ is Σ_P-measurable. □

5. Paving Nakano spaces by finite-dimensional sublattices

Let K be a fixed closed and bounded subset of $[1, \infty)$. Using the fact that any positively and contractively complemented sublattice of a Nakano space with essential range included in K is itself a Nakano space with essential range included in K, we will provide an intrinsic Banach lattice characterization of Nakano spaces with essential range included in K. This characterization is analogous to that of L_p-Banach spaces as Banach spaces that are paved by the spaces $(\ell_p^n \mid n \geq 1)$ almost isometrically (see e.g. [Lac74, p. 167]).

Given normalized Orlicz functions φ_k for $k = 1, \ldots, n$, we write $\ell_{(\varphi_k)}$ to denote the space $\mathbb{R}^n$ equipped with the modular $\Phi_{(\varphi_k)}((x_1, \ldots, x_n)) = \sum_{k=1}^n \varphi_k(x_k)$ and the norm

$$\|(x_1, \ldots, x_n)\|_{(\varphi_k)} = \inf\{\epsilon > 0 \mid \Phi_{(\varphi_k)}((x_1/\epsilon, \ldots, x_n/\epsilon)) \leq 1\}.$$

Then $\ell_{(\varphi_k)}$ is an n-dimensional Musielak-Orlicz space. If for each $k = 1, \ldots, n$, $\varphi_k(t) = |t|^{p_k}$ for some $p_k \geq 1$, then we write $\ell_{(p_k)}$ instead of $\ell_{(\varphi_k)}$. Note that $\ell_{(p_k)}$ is an n-dimensional Nakano space. In the lemma below we denote by $\mathrm{Id} : \ell_{(\varphi_k)} \to \ell_{(\phi_k)}$ the identity operator with domain $\ell_{(\varphi_k)}$ and range in $\ell_{(\phi_k)}$, and we write $\|\mathrm{Id} : \ell_{(\varphi_k)} \to \ell_{(\phi_k)}\|$ to denote its operator norm.

Lemma 5.1. *Let $n \in \mathbb{N}$. Let $1 \leq p_1 < p_2 < \cdots < p_n$ and for each $k = 1, \ldots, n$, let φ_k be normalized Orlicz functions such that for all $t \in (-1, 1)$ and all $k = 1, \ldots, n$, $|t|^{(1+\epsilon)p_k} < \varphi_k(t) \leq |t|^{p_k}$, for some $\epsilon > 0$ fixed. Then*

(i) $\|\mathrm{Id} : \ell_{(p_k)} \to \ell_{(\varphi_k)}\| = 1$;
(ii) $\|\mathrm{Id} : \ell_{(\varphi_k)} \to \ell_{((1+\epsilon)p_k)}\| = 1$;
(iii) $\|\mathrm{Id} : \ell_{((1+\epsilon)(p_k))} \to \ell_{(p_k)}\| \leq n^{\epsilon/(1+\epsilon)}$.

Proof. Let $(e_k)_{k=1}^n$ be the canonical basis of $\mathbb{R}^n$, and let x be an arbitrary element of $\mathbb{R}^n$. There exist $\alpha_k \in \mathbb{R}$ for $k = 1, 2, \ldots, n$, such that $x = \sum_{k=1}^n \alpha_k e_k$. Note that if $\|x\|_{(p_k)} \leq 1$, then $\Theta_{(p_k)}(x) \leq 1$, and so $|\alpha_k| \leq 1$ for all k, whence $\sum_{k=1}^n \varphi(\alpha_k) \leq \sum_{k=1}^n |\alpha_k|^{p_k} \leq 1$. By homogeneity, this implies that $\|x\|_{(\varphi_k)} \leq \|x\|_{(p_k)}$. This establishes (i). A nearly identical argument proves (ii). For (iii), observe that by the Hölder inequality for $l^{1+\epsilon}$ and $l^{(1+\epsilon)/\epsilon}$,

$$\sum |\alpha_k|^{p_k} \leq \left(\sum (|\alpha_k|^{p_k})^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \left(\sum 1^{\frac{1+\epsilon}{\epsilon}}\right)^{\frac{\epsilon}{1+\epsilon}},$$

i.e. $\Theta_{(p_k)}(\alpha) \leq \Theta_{((1+\epsilon)p_k)}(\alpha) n^{\frac{\epsilon}{1+\epsilon}}$. Hence if $\|\alpha\|_{((1+\epsilon)p_k)} = 1$ then $\Theta_{(p_k)}(\alpha) \leq n^{\frac{\epsilon}{1+\epsilon}}$ and by convexity

$$\Theta_{(p_k)}\left(n^{-\frac{\epsilon}{1+\epsilon}}\alpha\right) \leq 1.$$

This shows that $\|\alpha\|_{(p_k)} \leq n^{\frac{\epsilon}{1+\epsilon}}$. □

Lemma 5.2. *Let (Ω, Σ, μ) be a finite measure space. Let $A_1, \ldots, A_n$ be pairwise disjoint subsets of Ω that generate Σ and satisfy $\Omega = \cup_{i=1}^n A_i$. Let $\psi_1, \ldots, \psi_n$ be normalized Orlicz functions, and let ψ be a Musielak-Orlicz function on (Ω, Σ, μ) defined by setting*

$$\psi(t, \omega) = \sum_{i=1}^n \psi_i(t)\chi_{A_i}(\omega).$$

Then there exist normalized Orlicz functions $\varphi_1, \ldots, \varphi_n$ such that $L_\psi(\Omega, \Sigma, \mu)$ is isomodularly lattice-isometric to $\ell_{(\varphi_k)}$.

Proof. For $k = 1, \ldots, n$, define

$$\varphi_k(t) = \frac{\psi_k\left(t/\|\chi_{A_k}\|\right)}{\psi_k\left(1/\|\chi_{A_k}\|\right)}.$$

Let $T : L_\psi(\Omega, \Sigma, \mu) \to \ell_{(\varphi_k)}$ be defined by $T\left(\sum_{k=1}^n \alpha_k \chi_{A_k}/\|\chi_{A_k}\|\right) = \sum_{k=1}^n \alpha_k e_k$. If Ψ denotes the convex modular on $L_\psi(\Omega, \Sigma, \mu)$ defined by $\Psi(f) = \int_\Omega \psi(|f(\omega)|, \omega)\, d\mu(\omega)$, then

$$\Psi\left(\sum_{k=1}^n \alpha_k \frac{\chi_{A_k}}{\|\chi_{A_k}\|}\right) = \sum_{k=1}^n \varphi_k(\alpha_k),$$

whence T is an isomodular lattice-isomorphism. □

Remark 5.3. In the hypothesis of the lemma above, suppose $\psi_i = t \mapsto |t|^{p_i}$ for each $i = 1, \dots, n$. Then the change of density used to prove the lemma yields $\varphi_i = t \mapsto |t|^{p_i}$ for each $i = 1, \dots, n$, and this proves that $L_\psi(\Omega, \Sigma, \mu)$ is isomodularly lattice-isometric to $\ell_{(p_k)}$.

Definition. Let $\mathcal{F}$ be a class of finite dimensional normed lattices. A Banach lattice X is a *script-$(\lambda, \mathcal{F})$-lattice* if for every $\epsilon > 0$ and every finite system $(x_1, \dots, x_n)$ of positive disjoint elements of X there exists a finite-dimensional sublattice F of X which is $(\lambda+\epsilon)$-lattice isomorphic to a member of $\mathcal{F}$ and contains elements $x'_1, \dots, x'_n$ such that $\|x'_j - x_j\| \le \epsilon$ for all $j = 1, \dots n$.

If $\mathcal{F}$ is the class of finite-dimensional Nakano spaces with essential range included in the compact set K, we call X a *script-(λ, K)-Nakano space.*

Lemma 5.4. *A Banach lattice X is a script-$(\lambda, \mathcal{F})$-lattice if for every $\epsilon > 0$ and every finite-dimensional sublattice E of X there exists another finite-dimensional sublattice F of X and a vector lattice homomorphism $T : E \to F$ such that F is $(\lambda + \epsilon)$-lattice isomorphic to a member of $\mathcal{F}$ and $\|Tx - x\| \le \epsilon\|x\|$ for every $x \in E$.*

Proof. Let $(x_1, \dots, x_n)$ be a system of norm-one atoms of E and let $\delta > 0$. Let F be a finite-dimensional sublattice of X which is $(\lambda + \delta)$-lattice isomorphic to a member of $\mathcal{F}$, and which contains elements $x'_1, \dots, x'_n$ such that $\|x'_j - x_j\| \le \delta$ for all $j = 1, \dots, n$. Since for every j, $|x'_j| \in F$ and $||x'_j| - x_j| \le |x'_j - x_j|$, we may assume that the x'_j's are positive. Note, moreover, that

$$\||x'_j| - |x'_k|\| = \||x'_j| \wedge |x'_k|\| \le \|x'_j - x_j\| + \|x'_k - x_k\| \le 2\delta,$$

whence if we set

$$y_j = |x'_j| - |x'_j| \wedge \left(\sum_{k \ne j} |x'_k| \right)$$

we obtain positive disjoint elements of F such that $\|y_j - x'_j\| \le 2n\delta$ for $j = 1, \dots, n$. We may, therefore, assume without loss of generality that the x'_j's are positive and pairwise disjoint. Let $T : E \to F$ be the linear operator satisfying $Tx_j = x'_j$ for $j = 1, \dots, n$. Then if $x \in E$ and $x = \sum_{i=1}^n \alpha_i x_i$, we have that

$$\|Tx - x\| \le \sum_{i=1}^{n} |\alpha_i| \|Tx_i - x_i\| \le n(2n+1)\delta\|x\|$$

(since $|\alpha_i| \le \|x\|$, $i = 1, \dots, n$). Given any $\epsilon > 0$, an application of this argument with $\delta = \frac{\epsilon}{n(2n+1)}$ completes the proof. ☐

The following proposition generalizes Proposition 3.3 in [HR07]. Though the proof is very similar, we present it here for the reader's convenience.

Proposition 5.5. *Every script-$(\lambda, \mathcal{F})$-lattice X which does not contain c_0 is isometrically lattice isomorphic to a positively and contractively complemented sublattice of a Banach lattice which is λ-lattice isomorphic to an ultraproduct of members of $\mathcal{F}$.*

Proof. Since X does not contain c_0 as a sublattice, it is order continuous and therefore representable as a Köthe function space ([LT79, Th. 1.b.14]). Hence X is the closure of the union X_0 of some net (E_i) of its finite-dimensional sublattices (e.g., generated by simple functions). For each i and each $\epsilon > 0$, let $T_{i,\epsilon}$ be a lattice homomorphism from E_i into another finite-dimensional sublattice $F_{i,\epsilon}$ of X, itself $(\lambda + \epsilon)$-lattice isomorphic to a member of $\mathcal{F}$, such that

$$\|T_{i,\epsilon}x - x\| \le \epsilon\|x\|$$

for every $x \in E_i$. Let $\mathcal{U}$ be an ultrafilter on $I \times (0,\infty)$ containing all of the final sections $S_{i_0,\epsilon_0} = \{(i,\epsilon) \mid i > i_0, \epsilon < \epsilon_0\}$. Then the ultraproduct map $\widehat{T} : \prod_{i,\mathcal{U}} T_i$ is an isometric lattice embedding from $\widehat{E} = \prod_{i,\mathcal{U}} E_i$ into $\widehat{F} = \prod_{i,\mathcal{U}} F_{i,\epsilon}$, and this latter lattice is λ-lattice isomorphic to an ultraproduct of members of $\mathcal{F}$.

Moreover, $X_0 = \bigcup_i E_i$ embeds naturally (as a normed lattice) into $\widehat{E}$ by the diagonal embedding D_0 defined by $D_0(x) = (D_i(x))_{\mathcal{U}}$ where $D_i(x) = x$ if $x \in E_i$ and $D_i(x) = 0$ if $x \notin E_i$. The embedding D_0 extends in a unique way (by density and uniform continuity) to an embedding D of X into $\widehat{E}$. Then $\widehat{T}D$ is the desired embedding. If X is reflexive we may define positive contractive linear map $P : \widehat{F} \to X$ by $P([x_i]_{\mathcal{U}}) = w\text{-}\lim_{(i,\epsilon),\mathcal{U}} x_i$. Since for every $x \in X_0$ we have that $\lim_{(i,\epsilon),\mathcal{U}} \|T_{i,\epsilon}x - x\| = 0$, we have $P\widehat{T}Dx = x$ for every $x \in X_0$, and hence for every $x \in X$ by density and continuity, so P is a projection. In the case in which X is not reflexive we note that X is nevertheless a band in its bidual X^{**} and may consider the corresponding band projection π_X. Now, let $S : \widehat{F} \to X^{**}$ be the map defined by $S([x_i]_{\mathcal{U}}) = w^\star\text{-}\lim_{(i,\epsilon)} x_i$. Then $P = \pi_X S$ is the desired projection. ☐

Corollary 5.6. *Let $\mathcal{F}$ be a class of Banach lattices such that no ultraproduct of members of $\mathcal{F}$ contains c_0 as a sublattice. Then every class of Banach lattices which is closed under ultraproducts and positive contractive projections, and which contains $\mathcal{F}$, must also contain a lattice isometric copy of any script-$(1,\mathcal{F})$-lattice.*

By a result of Dacunha-Castelle ([DC72, Théorème 1]), the class $\mathcal{N}_{\subseteq K}$ consisting of all Banach lattices E such that E is isometrically lattice-isomorphic to a Nakano space with essential range included in K is closed under ultraproducts (see also Proposition 3.4.2 in [Poi06]). Moreover, the Corollary to 4.1 establishes the closure under positive contractive projections of this class.

Corollary 5.7. *Every script-$(1,K)$-Nakano space X is isometrically lattice isomorphic to a positively and contractively complemented sublattice of a Banach lattice which is lattice isomorphic to a Nakano space with essential range included in K.*

Proposition 5.8. *Every Nakano space X with essential range included in K is a script-$(1,K)$-Nakano space.*

Proof. Let $\epsilon > 0$ and let $(x_1,\dots,x_n)$ be a disjoint system of positive elements in X. By a change of density argument, we may assume that $x_i = \chi_{A_i}$ for some $A_i \in \Sigma$ with $\mu(A_i) < \infty$ $(i = 1,\dots,n)$. Let N be the least integer satisfying $(1+\epsilon)^N > P$, where P denotes the maximum of K.

Partition A_i into $A_i^{(1)},\dots,A_i^{(N)}$ so that for all $\omega \in A_i^{(j)}$, $(1+\epsilon)^{j-1} \le p(\omega) \le (1+\epsilon)^j$ for all $i = 1,\dots,n$. Let F be the sublattice of X generated by the finite set $\{\chi_{A_i^{(j)}} \mid (i,j) \in n \times N\}$. Then F is a Musielak-Orlicz space $L_\psi(\Omega_0,\Sigma_0,\mu_0)$ satisfying the following conditions: (1) $\Omega_0 = \cup_{i=1}^n A_i$ (so $\mu(\Omega_0) < \infty$); (2) Σ_0 is the σ-algebra generated by $\{A_i^{(j)} \mid (i,j) \in n \times N\}$; (3) $\mu_0 = \mu \restriction_{\Sigma_0}$; and (4) for all i,j there is a normalized Orlicz function $\psi_i^{(j)}$ in the closed convex hull of the functions $t \mapsto |t|^p$, for $p \in [(1+\epsilon)^{j-1},(1+\epsilon)^j)$, such that for all $t \in \mathbb{R}$ and for a.e. $\omega \in A_i^{(j)}$, $\psi(t,\omega) = \psi_i^{(j)}(t)$.

For $(i,j) \in n \times N$, set $p_i^{(j)} := \inf\{K \cap [(1+\epsilon)^{j-1},(1+\epsilon)^j)\}$. Define $\bar{p}(\cdot),\bar{\bar{p}}(\cdot) \in L_0(\Omega_0,\Sigma_0,\mu_0)$ by writing $\bar{p}(\omega) = \sum_{i,j} p_i^{(j)} \chi_{A_i^{(j)}}(\omega)$ and $\bar{\bar{p}}(\omega) = (1+\epsilon)\bar{p}(\omega)$.

Then $L_{\bar{p}(\cdot)}(\Omega_0,\Sigma_0,\mu_0)$ and $L_{\bar{\bar{p}}(\cdot)}(\Omega_0,\Sigma_0,\mu_0)$ are nN-dimensional Nakano spaces, and by Lemma 5.2 we have that $L_{\bar{p}(\cdot)}(\Omega_0,\Sigma_0,\mu_0)$ is isometrically lattice-isomorphic to $\ell_{(p_i^{(j)})}$ and $L_{\bar{\bar{p}}(\cdot)}(\Omega_0,\Sigma_0,\mu_0)$ is isometrically lattice-isomorphic to $\ell_{((1+\epsilon)p_i^{(j)})}$. By the same lemma, we can choose normalized Orlicz functions $\varphi_i^{(j)}$ (for $(i,j) \in n \times N$) such that $L_\psi(\Omega_0,\Sigma_0,\mu_0)$ is isometrically lattice-isomorphic to $\ell_{(\varphi_i^{(j)})}$. Moreover, since for each j, the Orlicz function $\psi_i^{(j)}$ is in the closed convex hull of the functions $t \mapsto |t|^p$ for $p \in [p_i^{(j)}, (1+\epsilon)p_i^{(j)})$, the same is true of $\varphi_i^{(j)}$. By Lemma 5.1, we obtain that $\|\mathrm{Id} : \ell_{(p_i^{(j)})} \to \ell_{(\varphi_i^{(j)})}\| = \|\mathrm{Id} : \ell_{(\varphi_i^{(j)})} \to \ell_{((1+\epsilon)p_i^{(j)})}\| = 1$ and that $\|\mathrm{Id} : \ell_{((1+\epsilon)p_i^{(j)})} \to \ell_{(p_i^{(j)})}\| \leq (nN)^{\frac{\epsilon}{1+\epsilon}}$. Since we have chosen N to satisfy $(1+\epsilon)^N \sim P$, we have that $N\log(1+\epsilon) \sim \log P$, whence

$$(nN)^{\frac{\epsilon}{1+\epsilon}} \sim \left[\frac{n\log P}{\log(1+\epsilon)}\right]^{\frac{\epsilon}{1+\epsilon}} \longrightarrow 1$$

as $\epsilon \to 0^+$. □

The upshot of these observations is the following

Theorem 5.9. *Let K be a closed and bounded subset of $[1,\infty)$. A Banach lattice X is a script-$(1,K)$-Nakano space if and only if it is isometrically lattice-isomorphic to a Nakano space with essential range included in K.*

6. A tighter characterization

We now turn our attention to the class $\mathcal{N}_K$ of Banach lattices X such that X is isometrically lattice-isomorphic to a Nakano space with essential range K.

Lemma 6.1. *Let $X := L_{p(\cdot)}$ be a Nakano space and assume that $(f,g) \in X^2$ is a system of two normalized disjoint functions generating isometrically $\ell^2_{(p,q)}$ as sublattice. Then $p(\omega) = p$ (resp. $= q$) almost everywhere on the support of f (resp. of g).*

Proof. By a change of density we may suppose that f and g are both indicator functions: $f = \chi_A$, $g = \chi_B$, with $A \cap B = \emptyset$. By the hypothesis we have:

$$\forall s,t \in [0,\infty), \quad s^p + t^q = 1 \implies \int_A s^{p(\omega)}\,d\mu(\omega) + \int_B t^{p(\omega)}\,d\mu(\omega) = 1.$$

For $x \geq 0$ let $F(x) = \int_A x^{p(\omega)/p}\,d\mu(\omega)$ and $G(x) = \int_B x^{p(\omega)/q}\,d\mu(\omega)$. We have then:

$$\forall x,y \in [0,\infty), \quad x+y=1 \Longrightarrow F(x)+G(y)=1.$$

From basic differentiation theorems for one-parameter integrals it follows that F and G are differentiable on $(0,+\infty)$ and

$$\forall x>0,\ F'(x) = \int_A \frac{p(\omega)}{p}\,x^{(p(\omega)/p)-1}\,d\mu(\omega) \text{ and } \quad G'(x) = \int_B \frac{p(\omega)}{q}\,x^{(p(\omega)/q)-1}\,d\mu(\omega).$$

Let $A_+ = \{\omega \in A \mid p(\omega) > p\}$, $A_0 = \{\omega \in A \mid p(\omega) = p\}$, $A_- = \{\omega \in A \mid p(\omega) < p\}$, then for $x \to 0$ we have:

$$\frac{1}{x}\int_{A_+\cup A_0} x^{p(\omega)/p}\,d\mu(\omega) \to \mu(A_0), \quad \frac{1}{x}\int_{A_-} x^{p(\omega)/p}\,d\mu(\omega) \to \begin{cases} +\infty, & \text{if } \mu(A_-) > 0,\\ 0, & \text{else.}\end{cases}$$

Hence $F'(0)$ exists if and only if $p(\omega) \geq p$ a.e. on A, and so $F'(0) = \mu(A_0)$. Similarly, $G'(0)$ exists if and only if $p(\omega) \geq q$ a. e. on B, and so $G'(0) = \mu(B_0)$, where $B_0 = \{\omega \in B \mid p(\omega) = q\}$.
Since $F(x) = 1 - G(1-x)$ and $G'(1)$ exists, we have that $F'(0)$ exists and $F'(0) = G'(1)$; similarly $G'(0)$ exists and $G'(0) = F'(1)$. Thus

$$\mu(A_0) = \int_B \frac{p(\omega)}{q}d\mu(\omega) \geq \mu(B) \text{ and } \mu(B_0) = \int_A \frac{p(\omega)}{p}d\mu(\omega) \geq \mu(A),$$

and since $\mu(A) = \mu(B) = 1$, it results $\mu(A_0) = 1$, $\mu(B_0) = 1$. Thus $A_0 = A$ and $B_0 = B$. □

Proposition 6.2. *Let $X := L_p(\Omega, \Sigma, \mu)$ be a Nakano space with compact essential range $K \subseteq [1, \infty)$ and such that $\dim(X) \geq 2$. The following statements are equivalent:*

(i) *For every $\epsilon > 0$, there exists a sublattice F of X which is $(1+\epsilon)$-lattice-isomorphic to the two-dimensional Nakano space $\ell^2_{(p,q)}$.*

(ii) *Some ultrapower $X_{\mathcal{U}}$ of X has a sublattice which is isometrically lattice-isomorphic to $\ell^2_{(p,q)}$.*

(iii) *$p \in K$.*

Proof. (i) ⇒ (ii): Let $I = (0, \infty)$ and let $\mathcal{U}$ be any ultrafilter on I containing the intervals $(0, \alpha)$ for $\alpha > 0$. Then

$$F = \{[(x_\epsilon)_{\epsilon>0}]_{\mathcal{U}} \mid \forall \delta > 0 \, \{\epsilon > 0 \mid \operatorname{dist}(x_\epsilon, F_\epsilon)\} \in \mathcal{U}\}$$

is a 2-dimensional sublattice of $X_{\mathcal{U}}$ which is isometrically lattice-isomorphic to $\ell^2_{(p,q)}$.

(ii) ⇒ (iii): Let $X_{\mathcal{U}} := L_{\widehat{p}(\cdot)}(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu})$ be an ultrapower of X and Y a sublattice of $X_{\mathcal{U}}$ which is isometrically lattice-isomorphic to $\ell^2_{(p,q)}$. The band $Y^{\perp\perp}$ generated by Y in $X_{\mathcal{U}}$ can be represented as a Nakano space $L_{\widehat{p}(\cdot)\restriction\Omega_0}(\Omega_0, \Sigma_0, \mu_0)$, where $\Omega_0 \subseteq \widehat{\Omega}$, $\Sigma_0 = \{A \cap \Omega_0 \mid A \in \widehat{\Sigma}\}$, and μ_0 is the restriction of $\widehat{\mu}$ to Ω_0. Let $y_1, y_2 \in Y$ satisfy $|y_1| \wedge |y_2| = 0$ and $\|y_1\| = \|y_2\| = 1$ and form an $\ell^2_{(p,q)}$-basis of Y (i.e., $\|\alpha y_1 + \beta y_2\| = \inf\{\epsilon > 0 \mid \left(\frac{\alpha}{\epsilon}\right)^p + \left(\frac{\beta}{\epsilon}\right)^q \leq 1\}$ for every $\alpha, \beta \in \mathbb{R}$). Then Lemma 6.1 ensures that $p(\cdot) = p$ on the support of y_1, and this in turn implies that p belongs to the essential range of $X_{\mathcal{U}}$. As we have remarked earlier, however, the essential range of $X_{\mathcal{U}}$ is a subset of the essential range K of X, and so $p \in K$.

(iii) ⇒ (i): The case in which $K = \{p\}$ is trivial, and so we may assume that there exist $p, q \in K$ with $p \neq q$. Let $\epsilon > 0$ be chosen such that the intervals $I_1 := ((1-\epsilon)p, (1+\epsilon)p)$ and $I_2 := ((1-\epsilon)q, (1+\epsilon)q)$ are disjoint, and let A_1 and A_2 be elements of Σ of positive measure such that $\{p(\omega) \mid \omega \in A_1\} \subseteq I_1$ and $\{p(\omega) \mid \omega \in A_1\} \subseteq I_2$. Consider the sublattice F of X generated by χ_{A_1} and χ_{A_2}. By the remark following Theorem 2.5, we may represent F as a two-dimensional Musielak-Orlicz space which is isometrically isomorphic, via Lemma 5.2, to $l^2_{(\varphi_1,\varphi_2)}$ for some normalized Orlicz functions φ_1, φ_2 in the convex hull of the functions $t \mapsto |t|^p$, for $p \in I_1 \cup I_2$. In fact, φ_1 satisfies the condition $|t|^{(1+\epsilon)p} \leq \varphi_1(t) \leq |t|^{(1-\epsilon)p}$ for all $t \in (-1, 1)$ and φ_2 satisfies the condition $|t|^{(1+\epsilon)q} \leq \varphi_1(t) \leq |t|^{(1-\epsilon)q}$ for all $t \in (-1, 1)$. An application of Lemma 5.1 yields that $\operatorname{dist}(\ell^2_{(p,q)}, F) \leq 2^{\frac{2\epsilon}{1+\epsilon}}$. The conclusion now follows by observing that $\lim_{\epsilon\to0+} 2^{\frac{2\epsilon}{1+\epsilon}} = 1$. □

Let K be a compact subset of $[1, \infty)$, and let X be a script-$(1, K)$-Nakano space. If there exists a countable dense subset D of K such that for every $p \in D$, for every $\epsilon > 0$, there exists a two-dimensional sublattice $F^{(p)}_\epsilon$ of X which is $(1+\epsilon)$-lattice isomorphic to $\ell^2_{(p,q)}$ for some $q \in K$, then we say that X is a *full* script-$(1, K)$-Nakano space.

Corollary 6.3. *Let K be a compact subset of $[1, \infty)$. A Banach lattice X is a full script-$(1, K)$-Nakano space if and only if it is isometrically lattice-isomorphic to a Nakano space with essential range K.*

Proof. If X Banach lattice which is isometrically lattice-isomorphic to a Nakano space with essential range K, then X is clearly a full script-$(1, K)$-Nakano space.

Conversely, if X is a full script-$(1, K)$-Nakano space, then by Proposition 5.9 it is isometrically lattice-isomorphic to a positively and contractively complemented sublattice Y of a Nakano space Z with essential range included in K. By Theorem 4.1 , Y is itself representable as a Nakano space with essential range $K' \subseteq K$. The fullness assumption moreover ensures by Proposition 6.2 that there is a countable dense subset D of K such that $D \subseteq K'$. Since K', being an essential range, is closed, we have that $K' = K$. □

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