1. Consider the statement

\[ |z - z_0| < \delta, \; |f(z) - f(z_0)| < \varepsilon. \]

Which of the following sentences correctly interprets the above statement?

(a) If the distance between \( z \) and \( z_0 \) is less than \( \delta \), then the distance between \( f(z) \) and \( f(z_0) \) is less than \( \varepsilon \).

(b) The absolute value of \( z \) minus the absolute value of \( z_0 \) is less than \( \delta \), and the absolute value of \( f(z) \) minus the absolute value of \( f(z_0) \) is less than \( \varepsilon \).

(c) The ball of radius \( \delta \) centered at \( z_0 \) is inside the ball of radius \( \varepsilon \) centered at \( f(z_0) \).

2. Form a team of two players, and play the *continuity* game, which is described as follows: player *Patient* claims that a given function \( f : \mathbb{C} \to \mathbb{C} \) is continuous at \( z_0 \), and player *Skeptic* claims this is not true. The two players alternate plays, with *Skeptic* playing first. *Skeptic*’s plays consist of positive real numbers, and *Patient*’s plays consist of positive real numbers as well. Suppose that during some round of play *Skeptic* plays \( \varepsilon > 0 \) and *Patient* plays \( \delta > 0 \). We say that *Patient wins* that round of play if the following condition is satisfied:

\[ |z - z_0| < \delta, \text{ we have that } |f(z) - f(z_0)| < \varepsilon. \]

Otherwise, we say that *Skeptic wins* that round of play. Happily for *Skeptic*, if she wins one round of play, then she wins the whole game. In order for *Patient* to win, she needs to win every single possible round of play. (*Patient must really be patient!*)

(a) Look at the following sample run of a few rounds of play: the function is \( f(z) = z \), and the value of \( z_0 \) is 0.

- *Skeptic* plays 2, and *Patient* plays 2.
- *Skeptic* plays 100, and *Patient* plays 100.
- *Skeptic* plays .001, and *Patient* plays .001.
- *Skeptic* plays \( \frac{1}{1000!} \) and *Patient* plays \( \frac{1}{1000!} \).

How would you score these rounds of play?

(b) In the same setting, what happens if *Skeptic* plays a fixed number \( \varepsilon > 0 \)? Can you find a value of \( \delta \) (preferably written in terms of \( \varepsilon \)) that ensures that *Patient wins*?
(c) In terms of the above described game, which of the following do you think is a good definition of continuity at $z_0$:

i. No matter what \textit{Skeptic} and \textit{Patient} play, \textit{Patient} always wins.


iii. Not so fast: in the example above, that was true, but in general, this may not be true. The point is that no matter what \textit{Skeptic} plays, \textit{Patient} can always cleverly choose her plays to win each round.

iv. Not so fast: the point is not that \textit{Patient} can win every single round of play, but that she can win almost all rounds of play, provided she is very clever. By “almost all” we mean that she can win all but finitely many rounds of play.

v. Not so fast: the point is not that \textit{Patient} can win almost all rounds of play, but that \textit{Patient} can, if she is clever, win infinitely many rounds of play.

(d) Let $f(z) = z^2$ and let $z_0 = 0$. For each of the following plays by \textit{Skeptic}, write down a winning play by \textit{Patient} if there is one:

i. \textit{Skeptic} plays 1.

ii. \textit{Skeptic} plays $\frac{1}{2}$.

iii. \textit{Skeptic} plays $\sqrt{2}$.

iv. \textit{Skeptic} plays $\frac{1}{10}$.

v. \textit{Skeptic} plays $\varepsilon > 0$.

(e) Let $g(z) = 2z^2 + 1$ and let $z_0 = 1$. Suppose that \textit{Skeptic} plays $\varepsilon > 0$. Write down a winning play by \textit{Patient}.

3. Suppose that $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous. Fix $z_0 \in \mathbb{C}$.

(a) Given $\varepsilon > 0$, which of the following statements can we make?

i. For any $\delta > 0$, if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$.

ii. There exists $\delta > 0$ such that if $|z - z_0| < \varepsilon$, then $|f(z) - f(z_0)| < \delta$.

iii. There exists $\delta > 0$ such that whenever $|z - z_0| < \delta$, $|f(z) - f(z_0)| < \varepsilon$.

iv. For any $\delta > 0$, $|f(z) - f(z_0)| < \delta$ whenever $|z - z_0| < \varepsilon$.

(b) Suppose that $g : \mathbb{C} \rightarrow \mathbb{C}$ is another continuous function. If for a given $\varepsilon > 0$, we know that whenever $|z - z_0| < \delta_1$, $|f(z) - f(z_0)| < \varepsilon$, do we also automatically know that whenever $|z - z_0| < \delta_1$, $|g(z) - g(z_0)| < \varepsilon$? If not, what do we know?

(c) Consider the composition $g \circ f : \mathbb{C} \rightarrow \mathbb{C}$, with $g$ as previously described. Suppose we aspire to prove that $g \circ f$ is continuous. Which of the following statements describes what we should do first?

i. There is nothing to be proved. It is obviously true that $g \circ f$ is continuous, because $f$ is continuous and $g$ is continuous. (Duh!)

ii. If we look up the definition of “continuous,” we realize that we need to prove that $g \circ f$ is continuous at each point $z_0$ in $\mathbb{C}$. So our first step should be to pick an arbitrary $z_0 \in \mathbb{C}$.

iii. We pick $\varepsilon > 0$.

iv. We graph the two functions, and then graph the composition, to get a better understanding of the situation.
v. We prove the statement by choosing simple examples, like \( f(z) = z^2 \), and \( g(z) = z^3 \). This is the easiest way to get anywhere.

(d) Given \( z_0, z_1 \in \mathbb{C} \) and \( \varepsilon > 0 \), suppose there exist \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that whenever \( |z - z_0| < \delta_1 \), \( |f(z) - f(z_0)| < \delta_2 \) and whenever \( |z - z_1| < \delta_2 \), \( |g(z) - g(z_1)| < \varepsilon \). Which of the following conditions would be sufficient to show that whenever \( |z - z_0| < \delta_1 \), \( |g(f(z)) - g(f(z_0))| < \varepsilon \)?

i. \( f(z_0) = z_1 \).
ii. \( g \circ f \) is continuous.
iii. No condition is sufficient: we need more information about the values of \( z_0 \) and \( z_1 \).

(e) Pick arbitrary \( z_0 \in \mathbb{C} \), and pick arbitrary \( \varepsilon > 0 \). Supposing \( f \) is continuous at \( z_0 \), which of the following statements are true:

i. There exists \( \delta_f > 0 \) such that whenever \( |z - z_0| < \delta_f \), \( |f(z) - f(z_0)| < \varepsilon \).
ii. There exists \( \delta_f > 0 \) such that whenever \( |z - z_0| < \delta_f \), \( |f(z) - f(z_0)| < \frac{\varepsilon}{2} \).
iii. There exists \( \delta_f > 0 \) such that whenever \( |z - z_0| < \delta_f \), \( |f(z) - f(z_0)| < \frac{\varepsilon}{3} \).
iv. All of the above.
v. None of the above.

(f) Pick arbitrary \( z_0 \in \mathbb{C} \), and arbitrary \( \varepsilon > 0 \). Supposing that \( g \) is continuous at \( f(z_0) \), determine whether the following statement is true or false:

There exists \( \delta_g > 0 \) such that whenever \( |z - f(z_0)| < \delta_g \), \( |g(z) - g(f(z_0))| < \varepsilon \).

(g) Pick arbitrary \( z_0 \in \mathbb{C} \). Suppose that \( g \) is continuous at \( f(z_0) \) and \( f \) is continuous at \( z_0 \). Let \( \varepsilon > 0 \) be arbitrary. Determine whether the following statement is true or false:

There exists \( \delta_g > 0 \) such that whenever \( |w - f(z_0)| < \delta_g \), \( |g(w) - g(f(z_0))| < \varepsilon \).

Next, determine whether the following statement is true or false:

There exists \( \delta_f > 0 \) such that whenever \( |z - z_0| < \delta_f \), \( |f(z) - f(z_0)| < \delta_g \).

Supposing, for the sake of argument, that the two previous answers are “true” and “true,” determine whether the following statement is true or false:

Whenever \( |z - z_0| < \delta_f \), \( |g(f(z)) - g(f(z_0))| < \varepsilon \).

4. Do Exercises 5, 10, and 11 in Section 17 of Brown and Churchill’s textbook.

5. Recall that

\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.
\]

Use this definition to prove that \( f(z) = z^2 \) is differentiable at \( z_0 \), for any \( z_0 \in \mathbb{C} \). (Compare your argument with that of Example 1 in Section 18 of Brown and Churchill’s textbook.)

6. Show that the function \( f(z) = |z|^2 \) is differentiable only at the point \( z_0 = 0 \). (Read Example 2 in Section 18 of Brown and Churchill’s textbook.)

7. Is it true that every continuous function is differentiable? Why or why not?

8. Is it true that every differentiable function is continuous? Why or why not?

9. Are the basic formulas of differentiation for functions of a complex variable the same as the basic formulas of differentiation for functions of a real variable?

10. Do Exercises 1 and 2 in Section 19 of Brown and Churchill’s textbook.