1. Prove that for any \( n \in \mathbb{N} \), \( n^5/5 + n^4/2 + n^3/3 - n/30 \) is an integer.

\textit{Solution.} We proceed by induction on \( n \). Let \( P(n) \) be the statement “\( n^5/5 + n^4/2 + n^3/3 - n/30 \) is an integer.” We first note that \( P(0) \) is trivially true, since 0 is an integer. We assume that the statement \( P(k) \) is true, and we attempt to show that \( P(k+1) \) is true. By expanding

\[
\frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30},
\]

we get

\[
\frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{k^4 + 4k^3 + 6k^2 + 4k + 1}{2} + \frac{k^3 + 3k^2 + 3k + 1}{3} - \frac{k+1}{30}.
\]

We recombine (to make use of the inductive hypothesis) terms to get

\[
\left[ \frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} \right] + \left[ (k^4 + 2k^3 + 2k^2 + k) + (2k^3 + 3k^2 + 2k) + (k^2 + k) \right].
\]

The first grouping is an integer by the inductive hypothesis, and the second grouping is an integer because it is a sum of integers. Thus, the proof follows by induction.

2. \textit{The Coin-Removal Problem:} Let a \textit{string} be a row of coins without gaps and without other coins beyond the ends. We write a string as a list of Hs and Ts. When we remove an H, we leave a gap (marked by a dot), and we flip all of the (at most two) coins next to it that remain. Thus \( HHT \) becomes \( T.H \) when we remove the \( H \) in the middle, and then we get \( T.. \) when we remove the new \( H \). Removing a coin from a string leaves two strings except when we remove the end. Can you guess what condition must a string satisfy in order for it to be possible to empty it (to remove all of its coins) using only repeated application of the rule above described? Use strong induction to prove that the condition you found is sufficient.

\textit{Solution.} We begin with a string of length \( n \). Examination of examples suggests that we can empty a string if and only if it has an odd number of Hs. We prove this by strong induction on \( n \). Let \( P(n) \) be the statement “We can empty a string consisting of \( n \) coins if and only if it has an odd number of Hs.” It is obvious that \( P(1) \) is true, since the only way in which we can empty a string
of length 1 is if the one coin in the string is an $H$. We assume that $P(i)$ is true for all $i \leq k$, and we attempt to show that $P(k+1)$ is true.

Consider a string $S$ of length $k+1$, for $k \geq 1$. In the case in which $S$ has an odd number of strings, let $X$ be its leftmost $H$. Remove $X$ and flip its neighbor or neighbors. The portion before $X$ now is empty or has one $H$, and the portion after $X$ has even weight, but if it is nonempty we flip its first element to obtain odd weight. Thus each remaining string is shorter than $S$ and has odd weight. By the induction hypothesis, each remaining string can be emptied, so $S$ can be emptied.

If $S$ has even weight, we show that removing any $H$ leaves a shorter nonempty string with even weight. For each $H$ in $S$, the number of other $H$s is odd. Thus there is an even number of them to one side and an odd number to the other. The side with an odd number is nonempty, and flipping its member next to the $H$ being removed makes it have an even number of $H$s. Thus for each $H$ we might remove, we leave a shorter nonempty string having an even number of $H$s. By the inductive hypothesis, this smaller string cannot be emptied, so $S$ cannot be emptied.

3. A Tournament of the Towns Problem: For any natural number $N$, prove the inequality

$$\sqrt{2\sqrt{3\sqrt{4\cdots\sqrt{(N-1)}\sqrt{N}}}} < 3.$$ 

**Solution.** This problem is too special. When we try to prove the inequality by induction, we quickly and certainly run into trouble. One way of escaping situations like this one is to consider a slightly more general problem, in the hope that this may lead us to greater insight. So we replace $2$ by $m$, and this makes the proof simpler. By specialization, we get the result. Fix a natural number $N \geq 2$. For $m \geq 2$, we let $P(m)$ be the statement "$\sqrt{m\sqrt{(m+1)\sqrt{\cdots\sqrt{N}}}} < m + 1$." We will prove that $P(m)$ is true for all $m = 2, 3, \ldots, N$ by a method we will call reverse induction. The base case is $P(N)$, which is obviously true, since $\sqrt{N} < N + 1$, for any natural number $N$. We suppose that $P(m+1)$ is true for $m \leq N-1$. We aim to show that $P(m)$ is true. By the inductive hypothesis, we have that

$$\sqrt{m\sqrt{(m+1)\sqrt{\cdots\sqrt{N}}}} < \sqrt{m(m+1)} < \sqrt{m^2 + 1} < \sqrt{m^2} = m,$$

and this establishes that $P(m)$ is indeed true. By reverse induction, $P(m)$ is true whenever $m \leq N$. Since $N$ was chosen arbitrarily, this is true for all $N \geq 2$.

4. Let $n$ points be selected along a circle and labeled by $a$ or $b$. Prove that there are at most $\left\lfloor (3n - 4)/2 \right\rfloor$ chords which join differently labeled points and which do not intersect inside the circle.

**Solution.** The result is obviously true for $n = 2$. Suppose we have already proved the theorem for $k < n$. Draw any diagonal connecting some $a$ to some nonadjacent $b$. (The case in which this is not possible is easy to handle.) The circle is split into two parts. One of the parts has $k$ points, and the other $n - k + 2$ points. We apply a suitable inductive hypothesis to both sides to get

$$\left\lfloor \frac{3k - 4}{2} \right\rfloor + \left\lfloor \frac{3(n - k + 2) - 4}{2} \right\rfloor + 1 \leq \left\lfloor \frac{3k - 4}{2} + \frac{3(n - k + 2) - 4}{2} + 1 \right\rfloor,$$
which is $\lfloor (3n - 4)/2 \rfloor$. Hence the theorem is valid for all $n$. (Question: What is this “suitable” inductive hypothesis?)

5. Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

**Solution.** Let $P(n)$ be the statement “$n$ amount of postage can be formed using just 4-cent and 5-cent stamps.” The base case is $P(12)$, which is true because $12 = 4 + 4 + 4$. It is useful to establish also $P(13)$, $P(14)$, and $P(15)$, by noticing that $13 = 5 + 4 + 4$, $14 = 5 + 5 + 4$, and $15 = 5 + 5 + 5$. Now, assume that $P(k)$ is true for all $k = 12, 13, \ldots, n - 1$, for $n \geq 16$. Consider $P(n)$. Since $n - 4 \in \{12, 13, \ldots, n - 1\}$, we know that $(n - 4)$ amount of postage can be formed by using just 4-cent and 5-cent stamps. By adding one 4-cent stamp, we obtain $n$ amount of postage just using 4-cent and 5-cent stamps.

6. Two players—Player I and Player II—alternately name dates. On each move, a player can increase the month or the day of the month but not both. The starting position is January 1, and the player who names December 31 wins. According to the rules, the first player can start by naming some day in January after the first or the first of some month after January. Determine a winning strategy for the Player I. (Hint: Use strong induction to describe the “winning dates.”)

**Solution.** Player I must always strive to get to one of the following dates: November 30, October 29, September 28, August 27, July 26, June 25, May 24, April 23, March 22, February 21, January 20. So Player I’s initial move should be January 20, and his strategy is pretty obvious.