1. Answer the following questions:

(a) Suppose $\Sigma$ is an infinite set of $L$-sentences, and that you know for a fact that every finite subset $\Sigma_0$ of $\Sigma$ is consistent. What theorem allows you to conclude that $\Sigma$ itself has a model?
A: _________________________________ (4 points)

(b) Suppose that $\mathcal{A}$ is an $L$-structure and you wish to find a “larger” structure $\mathcal{B}$ such that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. What theorem can you use?
A: _________________________________ (4 points)

(c) Suppose that $L$ is a countable language and that $\mathcal{A}$ is an $L$-structure that contains a given set $C$. Is there a countable elementary substructure $\mathcal{B}$ of $\mathcal{A}$ such that $C \subseteq B$? If so, what theorem guarantees this? If not, why not?
A: _________________________________ (4 points)

2. Show that the set of prime numbers is 0-definable in the semiring $\mathcal{N} = (\mathbb{N}, 0, 1, +, \cdot)$. (12 points)
3. Let $L = \{ \dot{f} \}$, where $\dot{f}$ is a unary function symbol. Consider the following four $L$-structures:

- $A_1 = (\mathbb{R}, f)$ where $f$ is given by $f(r) = r^2$;
- $A_2 = (\mathbb{N}, f)$ where $f$ is given by $f(n) = n + 1$;
- $A_3 = (\mathbb{N}, f)$ where $f$ is given by $f(n) = 2n$;
- $A_4 = (\mathbb{Z}, f)$ where $f$ is given by $f(n) = n + 1$.

For each $i \in \{1, 2, 3, 4\}$, write down an $L$-sentence $\sigma_i$ such that $\sigma_i$ is true in $A_i$ but false in all $A_j$ for $j \neq i$. (12 points)
4. Indicate a sentence in the language $L = \{<\}$ of totally ordered sets that is true in all finite nonempty totally ordered sets but false in some infinite linearly ordered set. (12 points)

5. Let $L = \{\hat{0}, \hat{1}, \hat{-}, \hat{+}, \hat{\cdot}\}$, where $\hat{0}$ and $\hat{1}$ are constant symbols, $\hat{-}$ is a unary function symbol, and $\hat{+}$ is a binary function symbol. Let $A_1 = (\mathbb{C}, 0, 1, -, +, \cdot)$, $A_2 = (\mathbb{R}, 0, 1, -, +, \cdot)$, and $A_3 = (\mathbb{Q}, 0, 1, -, +, \cdot)$. Write down an $L$-sentence $\sigma_1$ which is true in $A_1$ but false in $A_2$, and an $L$-sentence $\sigma_2$ which is true in $A_2$ but false in $A_3$. (12 points)
6. Let $L = \{<\}$ be the language of totally ordered sets. Suppose $\sigma$ is an $L$-sentence that is true in all infinite totally ordered sets. Show that there is a natural number $N$ such that $\sigma$ is also true in all finite totally ordered sets having more than $N$ elements. (12 points)
7. Let $L$ be a language and let $\sigma$ be an $L$-sentence. The *spectrum* of $\sigma$ is the set of cardinalities of finite models of $\sigma$, i.e. it is the set of all natural numbers $n$ for which $\sigma$ is satisfied in some structure the underlying set of which has exactly $n$ elements. Exhibit a language $L$ and a sentence $\sigma$ such that the spectrum of $\sigma$ is $\{1, 2, 3, 4\}$. (12 points)
8. Show that the theory of equivalence relations having infinitely many infinite equivalence classes is complete. (12 points)
9. Show that there exists a model $\mathcal{A} = (A, 0, 1, +, \cdot)$ of the theory of the structure $\mathcal{N} = (\mathbb{N}, 0, 1, +, \cdot)$ such that there exists $a \in A$ that is divisible by every $n \in \mathbb{N}$. (12 points)