1. Recall that a set $C$ in a vector space is said to be convex if for any $x, y \in C$ and any $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in C$. Show that a set $C$ in a vector space is convex if and only if $\sum \lambda_i x_i \in C$ whenever $x_1, \ldots, x_n \in C$ and $\lambda_1, \ldots, \lambda_n \geq 0$ satisfy $\sum \lambda_i = 1$. (20 points)
2. Let $1 \leq p \leq q \leq \infty$. Show that $\|x\|_{l_q} \leq \|x\|_{l_p}$ for any $x \in l_p$. (20 points)
3. A Banach space $X$ is separable if and only if $S_X$ is separable. (20 points)
4. Show that if $T : X \to Y$ is a bounded linear operator, then the kernel $\text{Ker}(T) = \{ x \in X \mid T(x) = 0 \}$ is a closed subspace of $X$. (20 points)
5. Let $X$ and $Y$ be normed spaces. Show that if $T : X \rightarrow Y$ is an isomorphism, then $T$ carries Cauchy sequences onto Cauchy sequences, and so if in such a situation $X$ is a Banach space, then $Y$ must also be a Banach space. (20 points)
6. (⋆) Let $\sum x_i$ be a series in a Banach space $X$, $x \in X$. The following are equivalent:

(a) For every $\epsilon > 0$, there is a finite set $F \subseteq \mathbb{N}$ such that

$$\left\| x - \sum_{i \in F'} x_i \right\| < \epsilon$$

whenever $F'$ is a finite set in $\mathbb{N}$ satisfying $F' \supseteq F$.

(b) If $\pi$ is any permutation of $\mathbb{N}$, then $\sum x_{\pi(i)} = x$.

Prove (only) that (a) implies (b). (20 points)