1. Let $X$ be a normed linear space. Prove that for any $x, y \in X$ we have $\|x\| - \|y\| \leq \|x - y\|$.

*Hint.* Triangle inequality, $\|x\| = \|(x - y) + y\|$.

2. Let $X$ be a normed linear space. Assume that for $x, y \in X$ we have $\|x + y\| = \|x\| + \|y\|$. Show that then $\|\alpha x + \beta y\| = \alpha \|x\| + \beta \|y\|$ for every $\alpha, \beta \geq 0$.

*Hint.* Assume $\alpha \geq \beta$. Write $\|\alpha x + \beta y\| = \|\alpha (x + y) - (\alpha - \beta)y\|$.

3. Recall that a set $C$ in a vector space is said to be convex if for any $x, y \in C$ and any $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in C$. Show that a set $C$ in a vector space is convex if and only if $\sum \lambda_i x_i \in C$ whenever $x_1, \ldots, x_n \in C$ and $\lambda_1, \ldots, \lambda_n \geq 0$ satisfy $\sum \lambda_i = 1$.

*Hint.* Note that $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = (\lambda_1 + \lambda_2) \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2 \right) + \lambda_3 x_3$.

4. Let $A$ and $B$ be two convex sets in a normed space $X$. Show that $\operatorname{conv}(A \cup B) = \{\lambda x + (1 - \lambda)y \mid x \in A, y \in B, \lambda \in [0, 1]\}$.

*Hint.* Show first that the set on the right-hand side is convex.

5. Let $1 \leq p \leq q \leq \infty$. Show that $\|x\|_{l_q} \leq \|x\|_{l_p}$ for any $x \in l_p$.

*Hint.* Consider the case $x = (x_i) \in l_p$ with $\|x\|_{l_p} = 1$.

6. Assume that $\int |f|^p \, dm < \infty$ for every $p \in [1, \infty)$. Prove that the map $J : [1, \infty) \to \mathbb{R}$ defined by $J(p) = \log \left( \int |f|^p \, dm \right)$ is a convex function.

*Hint.* Let $p < q$ and let $x \in (0, 1)$. Then $|f|^{xp} \in L^{1/x}$ and $|f|^{(1-x)q} \in L^{1/(1-x)}$. 

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Functional Analysis
Worksheet 2

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7. Let \( Y, Z \) be subspaces of a Banach space \( X \) such that \( Y \) is isomorphic to \( Z \). Are \( X/Y \) and \( X/Z \) isomorphic?

\[ \text{Hint. Let } X = l_2. \text{ Let } Y = \{ (0, x_2, x_3, \ldots) \mid x_i \in X \} \text{ and } Z = \{ (0, 0, x_3, x_4, \ldots) \mid x_i \in X \}. \]

8. Let \( X \) be the normed space obtained by taking \( c_0 \) with the norm \( \|x\|_0 = \sum 2^{-i}|x_i| \).

Show that \( X \) is not a Banach space.

\[ \text{Hint. The sequence } (1, 1, \ldots, 1, 0, \ldots) \text{ is Cauchy.} \]

9. A Banach space \( X \) is separable if and only if \( S_X \) is separable.

\[ \text{Hint. Too easy to give a hint.} \]

10. Let \( Y \) be a closed subspace of a Banach space \( X \). Show that if \( Y \) and \( X/Y \) are separable, then \( X \) itself is separable.

\[ \text{Hint. If } \{ \hat{x}_n \} \text{ is dense in } X/Y \text{ and } \{ x_n \} \text{ is dense in } Y, \text{ choose } y_n \in \hat{x}_n \text{ and consider } \{ y_n + x_k \mid n, k \in \mathbb{N} \}. \]

11. Find two subspaces \( F_1 \) and \( F_2 \) of a Banach space \( X \) such that \( F_1 \cap F_2 = \{ 0 \} \) and both \( F_1 \) and \( F_2 \) are dense in \( X \).

\[ \text{Hint. Let } X = C[0, 1], \text{ and let } F_1 \text{ be all polynomials on } [0, 1]. \text{ Find } F_2. \]