Functional Analysis
Worksheet 4

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1. Show that if X is an infinite-dimensional Banach space, then X admits a discontinuous linear functional. (Hint: Use the fact that every vector space has a basis.)

2. Let f be a linear functional on a Banach space X. Show that f is continuous if and only if $f^{-1}(0)$ is closed. Show also that if f is not continuous, then $f^{-1}(0)$ is dense in X.

3. If X is an infinite-dimensional Banach space, show that there are convex sets C_1 and C_2 such that $C_1 \cup C_2 = X$, $C_1 \cap C_2 = \emptyset$, and both C_1 and C_2 are dense in X.

4. Let X be a Banach space, and $f \in S_{X^*}$. Show that for every $x \in X$ we have $\operatorname{dist}(x, f^{-1}(0)) = |f(x)|$.

5. Let $(x_i)_{i=1}^n$ be a linearly independent set of vectors in a Banach space X and $(\alpha_i)_{i=1}^n$ be a finite set of real numbers. Show that there is $f \in X^*$ such that $f(x_i) = \alpha_i$ for $i = 1, \ldots, n$.

6. Let $(X, \|\cdot\|)$ be a Banach space. Show that $\mu_{B_X} = \|x\|$.

7. Let A, B be convex sets in a Banach space. Show that if $A \subseteq B$, then $\mu_B \leq \mu_A$. Show that $\mu_{cA}(x) = \frac{1}{c}\mu_A(x)$.

8. Let C be a convex subset of a Banach space X that contains a neighborhood of 0. Prove the following:

   (a) If C is also open, then $C = \{x : \mu_C(x) < 1\}$. If C is also closed, then $C = \{x : \mu_C(x) \leq 1\}$.

   (b) There is $c > 0$ such that $\mu_C(x) \leq c\|x\|$.

   (c) If C is moreover symmetric, then μ_C is a seminorm, that is, it is a homogeneous sublinear functional.

   (d) If C is moreover symmetric and bounded, then μ_C is a norm that is equivalent to $\|\cdot\|_X$.