1. Show that if $X$ is an infinite-dimensional Banach space, then $X$ admits a discontinuous linear functional. (Hint: Use the fact that every vector space has a basis.)

2. Let $f$ be a linear functional on a Banach space $X$. Show that $f$ is continuous if and only if $f^{-1}(0)$ is closed. Show also that if $f$ is not continuous, then $f^{-1}(0)$ is dense in $X$.

3. If $X$ is an infinite-dimensional Banach space, show that there are convex sets $C_1$ and $C_2$ such that $C_1 \cup C_2 = X$, $C_1 \cap C_2 = \emptyset$, and both $C_1$ and $C_2$ are dense in $X$.

4. Let $X$ be a Banach space, and $f \in S_{X^*}$. Show that for every $x \in X$ we have \[ \text{dist}(x, f^{-1}(0)) = |f(x)|. \]

5. Let $(x_i)_{i=1}^n$ be a linearly independent set of vectors in a Banach space $X$ and $(\alpha_i)_{i=1}^n$ be a finite set of real numbers. Show that there is $f \in X^*$ such that $f(x_i) = \alpha_i$ for $i = 1, \ldots, n$.

6. Let $(X, \| \cdot \|)$ be a Banach space. Show that $\mu_{B_X} = \|x\|$.

7. Let $A, B$ be convex sets in a Banach space. Show that if $A \subseteq B$, then $\mu_B \leq \mu_A$. Show that $\mu_{cA}(x) = \frac{1}{c} \mu_A(x)$.

8. Let $C$ be a convex subset of a Banach space $X$ that contains a neighborhood of 0. Prove the following:

   (a) If $C$ is also open, then $C = \{x : \mu_C(x) < 1\}$. If $C$ is also closed, then $C = \{x : \mu_C(x) \leq 1\}$.

   (b) There is $c > 0$ such that $\mu_C(x) \leq c\|x\|$.

   (c) If $C$ is moreover symmetric, then $\mu_C$ is a seminorm, that is, it is a homogeneous sublinear functional.

   (d) If $C$ is moreover symmetric and bounded, then $\mu_C$ is a norm that is equivalent to $\| \cdot \|_X$. 