Propose the sets of outcomes and the probability measures in the following situations.

1. A coin is tossed three times.
2. A die is rolled.
3. Two dice are rolled.
4. A die and a coin are thrown simultaneously.
5. A mug falls down from a table (luckily, an empty one).
6. From a pack of 52 cards, we draw 2.
7. A pack of six numbered cards is shuffled and the numbers are revealed one by one.
8. We play a series of chess games. The winner is the one who first scores three points, where one point is obtained for a single win and draws do not count. (⋆)
9. We keep throwing a coin until it lands heads up.
10. The temperature outdoors is measured. (⋆)
11. How long shall we wait for a bus at a stop? (⋆)
12. How many days does a letter posted in Salem take to reach Lynn?
13. What is the number of occurrences of the letter “e” in a 300-page long novel. (⋆)
14. How many road accidents happened today in your city? (⋆)
15. What time does your watch show now?
Solutions:

1. Let

\[ \Omega = \{(H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T)\}, \]

where \( H \) stands for heads, \( T \) for tails, and \( P(\{\omega\}) = 1/8 \) for each \( \omega \in \Omega \).

2. For one die: \( \Omega = \{1, 2, 3, 4, 5, 6\} \) with \( P(\{\omega\}) = 1/6 \) for each \( \omega \in \Omega \). For two dice: \( \Omega = \{(i, j) : i, j = 1, 2, \ldots, 6\} \), with \( P(\{\omega\}) = 1/36 \) for each \( \omega \in \Omega \).

3. \( \Omega = \{(i, a) : i = 1, 2, \ldots, 6, a \in \{H, T\}\}, \) and \( P(\{\omega\}) = 1/12 \) for each \( \omega \in \Omega \).

4. We take \( \Omega = \{Y, N\} \) if we are only interested in a Yes or No answer to the question of whether the mug will break. The number of pieces after landing can be represented by \( \Omega = \{1, 2, \ldots\} \). We can also take \( \Omega = \{U, D, R, L\} \) for the landing positions: up, down, handle to the right, or to the left, respectively. A more sophisticated choice is \( \Omega = \{U, D\} \cup [0, 2\pi] \) for a mug with no handle, but with a dot on its side. It may land up, down, or sideways, and then roll, in which case we find the angle between the floor and the straight line containing the radius determined by the dot. In each case, the choice of a probability measure depends on the physical properties of the mug and the floor.

5. There are two possibilities:

   (a) We draw two cards at once; \( \Omega \) consists of all two-element subsets \( \{c_1, c_2\} \) of \( C \).

   (b) We draw the second card after having returned the first one to the pack. Then \( \Omega \) consists of all ordered pairs \( (c_1, c_2) \) of cards \( c_1, c_2 \in C \).

   Unless the cards are drawn by a magician, we can take the probability measure \( P(\{\omega\}) = 1/n \), where \( n \) denotes the number of elements of \( \Omega \).

6. \( \Omega \) consists of all permutations of the set \( \{1, \ldots, 6\} \). Each outcome is equally likely, so we take the uniform probability \( P(\{\omega\}) = 1/6! \) for each \( \omega \in \Omega \).
7. The outcomes are all the three-, four-, and five-element sequences of letters W or L, each containing exactly three letters of one kind (20 possibilities). The uniform probability does not seem appropriate. Everything depends on the probability of winning a single game. Even if we assume that it does not change from game to game and is known, it is not obvious what the measure should be.

8. \( \Omega = \{0, 1, 2, \ldots \} \), the number of tails before the first head appears. \( P(\{0\}) = 1/2, P(\{1\}) = 1/4, P(\{2\}) = 1/8, \) and so on. We could also add the outcome corresponding to the fact that the head never appears. This is theoretically possible, so put \( \Omega' = \Omega \cup \{\infty\} \). The above assignment, \( P(\{\omega\}) = 1/2^{\omega+1} \) remains valid. Because \( P(\Omega) = 1 \), this leaves us with only one possible choice: \( P(\{\infty\}) = 0 \). Here, we have used the fact that \( 1/2 + 1/4 + \cdots + 1/2^n + \cdots = 1 \).

9. An idealistic (but mathematically convenient) approach is to assume that the temperature is a real number, so \( \Omega = (-\infty, \infty) \), with probability measure given by a density supplied by the meteorological office. This density depends on the country in which you live. If the accuracy of meteorological data is within 1°, we may prefer \( \Omega = \mathbb{Z} \).

10. If, for example, the line is serviced by six buses and each takes 60 minutes for a complete round, then \( \Omega = [0, 10] \) with uniform density seems reasonable. If the bus has a timetable, a density with a peak at each appropriate time may be better.

11. USPS-endorsed solution: \( \Omega = \{1\} \), if you use a first-class stamp and it is not snowing!

12. Oddly enough, there is a 300+-page French novel in which the letter e never appears.

13. It may be convenient to take \( \Omega = \{0, 1, 2, \ldots \} \) with a special probability measure \( P(\{n\}) = e^{-\lambda n} \frac{\lambda^n}{n!} \) for each \( n = 0, 1, \ldots \). This is called the Poisson measure with parameter \( \lambda > 0 \). We won’t get into this!

14. If it is a watch with a simple digital display, we can take \( \Omega_{\text{hours}} = \{1, \ldots, 12\}, \Omega_{\text{minutes}} = \{1, \ldots, 59\}, \Omega_{\text{seconds}} = \{1, \ldots, 59\}, \) and \( \Omega = \{(h, m, s) : h \in \Omega_{\text{hours}}, m \in \Omega_{\text{minutes}}, s \in \Omega_{\text{seconds}}\} \). Some hours are more likely than others (why??), but all values of \( m \) and \( s \) seem equally probable.