

# Research Statement

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## 1 Historical background

My research interests fit broadly in the intersection between mathematical logic and functional analysis. This intersection has been the locus of important activity since the 1960's, when Bretagnolle, Dacunha-Castelle, and Krivine introduced the concept of Banach space ultraproduct [3], and Luxemburg defined nonstandard hulls [8]. Both constructions went on to become useful tools in Banach space theory, and other successful adaptations to analysis of ideas and techniques from mathematical logic followed. Their consequences included the following striking results:

1. Tsirelson's example of a Banach space not containing  $l_p$  or  $c_0$ , inspired by the method of forcing.
2. Krivine's Theorem<sup>1</sup>, proved using ultraproducts and compactness.
3. Krivine-Maurey's result that every stable Banach space contains almost isometrically a copy of  $l_p$ , using the model-theoretic concept of quantifier-free stability.
4. Bourgain-Rosenthal-Schechtman's result that there are uncountably many complemented subspaces of  $L_p$ , using ordinal ranks of a type that are used in model theory.
5. Gowers' dichotomy, proved using Gowers' block Ramsey Theorem, inspired in turn by the Galvin-Prikry proof that every Borel set is Ramsey.

In [6], Henson worked out a logical framework (see also [7]), the positive bounded logic with approximate semantics, to provide a model theoretic foundation for the uses of Banach space ultraproducts, and recently Berenstein, Ben-Yaacov, Henson, and Usvyatsov have developed a newer framework, continuous logic for metric structures, which provides an equivalent background for the ultraproduct construction, and which is easier to use in applications. Both settings render transparent the relationship between many of the key concepts used in Banach space theory and their analogues in mathematical logic, and constitute a point of departure for new research on the model theory of structures in analysis. The fact that some of the concepts and techniques from model theory have been incorporated so successfully to Banach space theory suggests that the relationship between the two fields is robust, and it motivates further inquiry.

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<sup>1</sup>Krivine's Theorem states that whenever  $(x_n)$  is a sequence with infinite-dimensional span in a Banach space, then the unit vector basis of  $c_0$  or  $l_p$  (for some  $1 < p < \infty$ ) is block finitely represented in  $(x_n)$ . Because it is easy to show that  $l_2$  is finitely represented in  $c_0$  and in  $l_p$  (for every  $1 < p < \infty$ ), Dvoretzky's Theorem that  $l_2$  is finitely represented in every Banach space is an easy corollary of Krivine's Theorem.

## 2 A few preliminary remarks

Two fundamental concepts of model theory are quantifier elimination and stability. A theory is said to admit quantifier elimination if for every formula  $\phi$  there is a quantifier-free formula  $\psi$  equivalent to it. A typical example illustrating the idea of quantifier elimination is the following: let  $\phi(a, b, c)$  be the formula  $\exists x ax^2 + bx + c = 0$ . Then, in  $\mathbb{R}$ ,  $\phi(a, b, c)$  is equivalent to the formula  $(a \neq 0) \wedge b^2 - 4ac \geq 0) \vee (a = 0 \wedge (b \neq 0 \vee c = 0))$ . One of the upshots of having quantifier elimination is that the study of definable sets (sets described by formulas) is made considerably simpler. Quantifier elimination has a slightly different definition in the setting of continuous logic for metric structures, but the intuition is the same: a theory in continuous logic is said to admit quantifier elimination if every formula can be uniformly approximated by quantifier free formulas.

A *type*, in model theory, is a consistent set of formulas. A type is said to be *complete* if it is maximally consistent. A complete (first-order) theory with infinite models is called  $\kappa$ -*stable* (for some infinite cardinal  $\kappa$ ) if it has no more than  $\kappa$ -many complete types over any parameter-set of size  $\kappa$ . A theory is said to be *stable* if it is  $\kappa$ -stable for some infinite cardinal  $\kappa$ . In continuous logic for metric structures, this concept of stability carries over nicely<sup>2</sup>. In Banach space theory, types were initially defined as follows. If  $X$  is a Banach space and  $a \in X$ , the *type of  $a$*  is the function  $\tau_a : X \rightarrow \mathbb{R}$  defined by  $\tau_a(x) = \|a + x\|$ . It turns out that this notion of type corresponds with the model-theoretic notion of quantifier-free type. A Banach space  $X$  is said to be *Krivine-Maurey stable* if whenever  $(a_m)$  and  $(b_n)$  are bounded sequences in  $X$  and  $\mathcal{U}, \mathcal{V}$  are ultrafilters on  $\mathbb{N}$ ,

$$\lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} \tau_{a_m}(b_n) = \lim_{n, \mathcal{V}} \lim_{m, \mathcal{U}} \tau_{a_m}(b_n).$$

Krivine-Maurey stability corresponds exactly with model-theoretic quantifier-free stability in the language of Banach spaces. The importance and utility of the concept of Krivine-Maurey stability in the geometry of Banach spaces indicates that it makes sense to study full model-theoretic stability (and its consequences) in the geometry of specific structures in functional analysis.

## 3 $L_p(\mu)$ -spaces

Using the positive bounded logic developed by Henson, the model theory of classical  $L_p(\mu)$ -spaces has been studied thoroughly (see [2]), and is now well understood. The class of  $L_p(\mu)$ -spaces over atomless measure spaces is axiomatizable by positive bounded sentences, and admits quantifier elimination, in the language  $\mathcal{L}$  of Banach lattices. Moreover, this class is model-theoretically stable relative to  $\mathcal{L}$ , and the model-theoretic independence relation that arises from stability has been characterized in terms of important concepts from probability theory.

## 4 Musielak-Orlicz spaces

A function  $\varphi : \mathbb{R} \rightarrow [0, \infty)$  is said to be an *Orlicz function* if it is convex, continuous, even, and  $\varphi(0) = 0$ , and it is said to be *normalized* if additionally  $\varphi(1) = 1$ . Henceforth, we will convene that all Orlicz functions are normalized. (The typical example of an Orlicz function is the function  $t \mapsto |t|^p$ , where  $1 \leq p < \infty$ .) Let  $(\Omega, \Sigma, \mu)$  be a measure space. A *Musielak-Orlicz function* on  $(\Omega, \Sigma, \mu)$  is a function  $\psi : \mathbb{R} \times \Omega \rightarrow [0, \infty)$  such that the following two conditions are satisfied:

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<sup>2</sup>In fact, stability in this setting is relative to a topology; in this narrative, we implicitly work with the topology induced by the metric.

(A)  $\psi(t, \cdot) \in L_0(\Omega, \Sigma, \mu)$  for every  $t \in \mathbb{R}$ .

(B)  $\psi(\cdot, \omega)$  is an Orlicz function for all  $\omega \in \Omega$ .

Given a measure space  $(\Omega, \Sigma, \mu)$  and a Musielak-Orlicz function  $\psi$  on it, we define its associated *modular*  $\Psi : L_0(\Omega, \Sigma, \mu) \rightarrow [0, \infty]$  by

$$\Psi(f) = \int_{\Omega} \psi(f(\omega), \omega) d\mu(\omega)$$

and set

$$\|f\|_{\psi} = \inf \left\{ \alpha \in [0, \infty) \mid \Psi\left(\frac{f}{\alpha}\right) \leq 1 \right\}.$$

Let

$$L_{\psi}(\Omega, \Sigma, \mu) = \{f \in L_0(\Omega, \Sigma, \mu) \mid \|f\|_{\psi} < \infty\}.$$

Then  $L_{\psi}(\Omega, \Sigma, \mu)$  is a *Musielak-Orlicz space*. The norm  $\|\cdot\|_{\psi}$  is usually called the *Luxemburg norm* in the literature on Musielak-Orlicz spaces. In the case in which

$$\{\psi(\omega, \cdot) \mid \omega \in \Omega\} = \{\varphi\}$$

for some fixed Orlicz function  $\varphi$ , then we call the associated Musielak-Orlicz space an *Orlicz space* and we denote it by

$$L_{\varphi}(\Omega, \Sigma, \mu).$$

Observe that  $L_p(\mu)$ -spaces are examples of Orlicz spaces. It is sensible then to ask whether we can generalize what we know about the model theory of  $L_p(\mu)$ -spaces to Orlicz spaces. A good starting point is to pin down the class of all structures elementarily equivalent to a given Orlicz space. It turns out that a general ultrapower of an Orlicz space need not be representable as an Orlicz space, and so the class of Orlicz spaces is not the right setting to which it is sensible to generalize what we know about  $L_p(\mu)$ -spaces. However, by the work of Dacunha-Castelle [4], we know that certain classes of Musielak-Orlicz spaces are indeed closed under taking ultraproducts, and so Musielak-Orlicz spaces are better candidates to carry out our investigations.

Let  $k > 0$ . We say that a Musielak-Orlicz space  $L_{\psi}(\Omega, \Sigma, \mu)$  satisfies the  $\Delta_2^k$ -condition if  $\Psi(2f) \leq k\Psi(f)$  for all  $f \in L_{\psi}(\Omega, \Sigma, \mu)$ . Let  $\mathcal{MO}_k$  denote the class of all Musielak-Orlicz spaces satisfying the  $\Delta_2^k$ -condition.

For  $k > 0$ , let  $\mathcal{F}_k$  denote the class of all Orlicz functions satisfying the  $\Delta_2^k$ -condition, and equip  $\mathcal{F}_k$  it with the topology of pointwise convergence. Then  $\mathcal{F}_k$  is a compact subspace of  $\mathbb{R}^{\mathbb{R}}$ . We say that a set  $D$  of normalized Orlicz functions is a *Dacunha-Castelle set* if  $D$  is a closed (hence compact) subset of  $\mathcal{F}_k$  (for some  $k$ ) which is *closed under dilations*, i.e. for any  $\varphi \in D$  and any  $a > 0$ ,  $t \mapsto \frac{\varphi(at)}{\varphi(a)}$  belongs to  $D$ . Let  $H_D$  denote the closure of the convex hull of  $D$  in  $\mathcal{F}_k$ . Dacunha-Castelle proved in [4] the following:

**Theorem 1 (Dacunha-Castelle).** *Let  $D$  be a Dacunha-Castelle set. Then*

$$\mathcal{MO}_D = \{L_{\psi}(\Omega, \Sigma, \mu) \in \mathcal{MO}_k \mid \forall \omega \in \Omega \quad \psi(\cdot, \omega) \in D\}$$

*is closed under ultraproducts.*

A consequence of this, appearing in my thesis, is:

**Corollary 2.** *If  $D$  is a Dacunha-Castelle set, then  $\mathcal{MO}_{H_D}$  is a universally axiomatizable class of Banach lattices.*

Another result in this direction is the following theorem from my thesis:

**Theorem 3.** *Any  $L_\psi(\Omega, \Sigma, \mu)$  in  $\mathcal{MO}_{H_D}$  can be embedded as a sublattice of an element of the class  $\mathcal{MO}_D$ .*

An application of the Downward-Löwenheim-Skolem Theorem improves the above theorem:

**Corollary 4.** *Any separable  $L_\psi(\Omega, \Sigma, \mu)$  in  $\mathcal{MO}_{H_D}$  can be embedded as a sublattice of a separable element of the class  $\mathcal{MO}_D$ .*

## 5 Nakano spaces

Let  $(\Omega, \Sigma, \mu)$  be a measure space, and let  $L_0(\Omega, \Sigma, \mu)$  denote the partially ordered linear space of all measurable functions from  $\Omega$  into  $\mathbb{R}$ , with two functions identified when they agree except on a set of measure 0. Consider an essentially bounded random exponent  $p(\cdot) \in L_0(\Omega, \Sigma, \mu)$ ; that is, assume that  $p(\cdot)$  satisfies  $\text{ess inf } p(\omega) \geq 1$  and  $\text{ess sup } p(\omega) \leq B < \infty$  for some  $B > 0$  fixed. Thus we are allowing the exponent  $p(\omega)$  to vary measurably as  $\omega$  varies over  $\Omega$ , requiring that  $1 \leq p(\omega) \leq B < \infty$  for almost all  $\omega$ .

Define  $\Theta_{p(\cdot)} : L_0(\Omega, \Sigma, \mu) \rightarrow [0, \infty]$  by setting

$$\Theta_{p(\cdot)}(f) := \int_{\Omega} |f(\omega)|^{p(\omega)} d\mu(\omega).$$

Let  $\|\cdot\|_{p(\cdot)} : L_0(\Omega, \Sigma, \mu) \rightarrow [0, \infty]$  be defined by

$$\|f\|_{p(\cdot)} := \inf\{\epsilon > 0 \mid \Theta_{p(\cdot)}(f/\epsilon) \leq 1\}.$$

Let

$$L_{p(\cdot)}(\Omega, \Sigma, \mu) := \{f \in L_0(\Omega, \Sigma, \mu) \mid \|f\|_{p(\cdot)} < \infty\}.$$

We say that  $L_{p(\cdot)}(\Omega, \Sigma, \mu)$  thus defined is a *Nakano space*. With the norm  $\|\cdot\|_{p(\cdot)}$  and with the pointwise partial ordering and vector space structure that it inherits from  $L_0(\Omega, \Sigma, \mu)$ , the associated Nakano space is a Banach lattice. Notice that Nakano spaces are interesting in two different ways: they constitute generalizations of  $L_p(\mu)$ -spaces that are not rearrangement invariant, and they constitute the simplest nontrivial example of Musielak-Orlicz spaces (the Musielak-Orlicz function  $\psi$  is given by  $\psi(t, \omega) = |t|^{p(\omega)}$ ).

For  $p(\cdot) \in L_0(\Omega, \Sigma, \mu)$  given, we define its *essential range* to be the set

$$\mathcal{R}_{p(\cdot)} = \{q \in [1, \infty) \mid \forall \epsilon > 0 \quad \mu(\{\omega \in \Omega \mid |p(\omega) - q| < \epsilon\}) > 0\}.$$

Let  $K$  be a compact subset of  $(1, \infty)$ . Let

$$\mathcal{N}_K = \{L_{p(\cdot)}(\Omega, \Sigma, \mu) \mid \mathcal{R}_{p(\cdot)} = K, \quad (\Omega, \Sigma, \mu) \text{ atomless}\}.$$

## 6 Results and questions

The following results (along with the ones mentioned in the preceding section) have been established in my thesis:

**Theorem 5.** *The class  $\mathcal{N}_K$  is axiomatizable in the language of Banach lattices.*

**Theorem 6.** *Consider the class  $\mathcal{N}_K^{\text{mod}}$  of Nakano spaces in  $\mathcal{N}_K$  as Banach lattices with the associated modular  $\Theta_{p(\cdot)}$ . This class is also axiomatizable, and it admits quantifier elimination. Moreover, its approximate theory is complete; that is, any two Nakano spaces in  $\mathcal{N}_K^{\text{mod}}$  are elementarily equivalent.*

**Theorem 7.** *The class  $\mathcal{N}_K^{\text{mod}}$  is model-theoretically stable.*

The proofs of the above theorems are more delicate than the proofs for the corresponding facts about  $L_p(\mu)$ -spaces. Some questions remain:

**Question 1.** *Are the modulars mentioned above uniformly definable in the Banach lattice structure of the Nakano spaces in  $\mathcal{N}_K$ ?*

**Question 2.** *What is the independence relation that arises from the stability of the class  $\mathcal{N}_K^{\text{mod}}$ ?*

**Conjecture.** *We have required that  $\inf K > 1$  to avoid certain complications that arise when considering the more natural case  $\inf K \geq 1$ . The effect of this is that all Nakano spaces in the class  $\mathcal{N}_K$  are reflexive, and that plays a technical role in the current proofs. However, it should be possible to refine the existing arguments to allow  $\inf K \geq 1$ . The natural conjecture is that the above theorems remain true for this more general setting. It is also important to reformulate the previous questions in this more general setting, should the conjecture turn out to be true.*

**Question 3.** *So far, in studying Nakano spaces, I have only been able to achieve  $\kappa$ -stability for  $\kappa$  larger than or equal to the cardinality of the continuum, and in general this is as good as the result can get. It is interesting to ask what minimal restrictions one can impose on the behaviour of the random exponent  $p(\cdot)$  of a Nakano space in order to achieve  $\omega$ -stability.*

## 7 Future plans

Aside from answering open questions about the class of Nakano spaces, there are several directions in which my research can go. Right now is a moment of extraordinary novelty, activity, and excitement in the area, and little is known about the model theory of many interesting classes of structures in analysis.

1. Before settling on Nakano spaces, I attempted to tackle the more general problem of understanding the model theory of Musielak-Orlicz spaces (of which Nakano spaces are a paradigmatic example), but I ran into some difficulties. I am currently thinking about ways of overcoming these difficulties, in light of the progress made on Nakano spaces.
2. It is open whether  $E(X)$  is Krivine-Maurey stable whenever  $E$  is a Krivine-Maurey stable Köthe function space and  $X$  is a Krivine-Maurey stable Banach space. The implication is known to be true when  $E$  is either atomic (a sequence space) or a rearrangement invariant space. It is a natural weakening of the question to ask if  $E(X)$  must be stable if  $X$  is a stable Banach space and we make the stronger assumption that  $E$  is quantifier-free stable as

a Banach lattice. Further if  $X$  is also quantifier-free stable as a Banach lattice, we can ask if this property also holds of  $E(X)$ . The approach of Hensgen in [5] may be useful here. This problem has been suggested to me by Y. Raynaud, and it fits very well within my area of expertise and interest.

3. In a more abstract direction, the development of continuous logic for metric structures has given rise to many interesting questions. Ben-Yaacov has recently proved the analogue of Morley's Theorem for metric structures [1]. (If a complete theory of a metric structure for a countable language is categorical in one uncountable power, then it is categorical in all uncountable powers.) An interesting and challenging open question is whether the only examples of uncountably categorical metric structures are closely based on Hilbert space.
4. In [9], Odell has remarked that all the counterexamples to Banach's classical conjecture that every Banach space contains a copy of  $l_p$  (for some  $1 \leq p < \infty$ ) or  $c_0$  rely on norms *implicitly* defined by some fixed point argument.

**Question 4 (Odell).** *Formulate a condition which interprets the statement "the norm  $\|\cdot\|$  on  $c_{00}$  is explicit." Then prove that if  $X$  is the completion of  $c_{00}$  under an explicit norm, then  $X$  contains (almost isometric) copies of  $c_0$  or  $l_p$  for some  $1 \leq p < \infty$ .*

## References

- [1] I. Ben-Yaacov. Uncountable dense categoricity in cats. *Journal of Symbolic Logic*, 70(3):829–860, 2005.
- [2] I. Ben-Yaacov, A. Berenstein, and C. W. Henson. Model theoretic independence in the Banach lattices  $L_p(\mu)$ .
- [3] J. Bretagnolle, D. Dacunha-Castelle, and J.-L. Krivine. Lois stables et espaces  $L^p$ . *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 2:231–259, 1965/1966.
- [4] D. Dacunha-Castelle. Sur un théorème de J.-L. Krivine concernant la caractérisation des classes d'espaces isomorphes à des espaces d'Orlicz généralisés et des classes voisines. *Israel J. Math.*, 13:261–276, 1972.
- [5] W. Hensgen. Some properties of the vector-valued ideal space  $E(X)$  derived from those of  $E$  and  $X$ . *Collect. Math.*, 43:1–13, 1992.
- [6] C. W. Henson. Nonstandard hulls of Banach spaces. *Israel J. of Math.*, 25:108–144, 1976.
- [7] C. W. Henson and J. Iovino. Ultraproducts in Analysis. In *Analysis and Logic*, volume 262 of *Lecture Notes Series*, pages 1–113. Cambridge University Press, 2003.
- [8] W. Luxemburg. A general theory of monads. In *Applications of Model Theory to Algebra, Analysis, and Probability*, pages 18–86. Holt, Rinehart and Winston, 1969.
- [9] E. Odell. On Subspaces, Asymptotic Structure, and Distortion of Banach Spaces; Connections with Logic. In *Analysis and Logic*, volume 262 of *Lecture Notes Series*, pages 191–267. Cambridge University Press, 2003.